

Large p -groups actions with $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$.

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Abstract

Let k be an algebraically closed field of characteristic $p > 0$ and C a connected nonsingular projective curve over k with genus $g \geq 2$. Let (C, G) be a "big action", i.e. a pair (C, G) where G is a p -subgroup of the k -automorphism group of C such that $\frac{|G|}{g} > \frac{2p}{p-1}$. We first study finiteness results on the values taken by the quotient $\frac{|G|}{g^2}$ when (C, G) runs over the big actions satisfying $\frac{|G|}{g^2} \geq M$, for a given positive real $M > 0$. Then, we exhibit a classification and a parametrization of such big actions when $M = \frac{4}{(p^2-1)^2}$.

1 Introduction.

Setting. Let k be an algebraically closed field of positive characteristic $p > 0$ and C a connected nonsingular projective curve over k , with genus $g \geq 2$. As in characteristic zero, the k -automorphism group of the curve C , $\text{Aut}_k(C)$, is a finite group whose order is bounded from above by a polynomial in g (cf. [St73] and [Sin74]). But, contrary to the case of characteristic zero, the bound is no more linear but biquadratic, namely: $|\text{Aut}_k(C)| \leq 16g^4$, except for the Hermitian curves: $W^q + W = X^{1+q}$, with $q = p^n$ (cf. [St73]). The difference is due to the appearance of wild ramification. More precisely, let G be a subgroup of $\text{Aut}_k(C)$. If the order of G is prime to p , then the Hurwitz bound still holds, i.e. $|G| \leq 84(g-1)$. Now, if G is a p -Sylow subgroup of $\text{Aut}_k(C)$, Nakajima (cf. [Na87]) proves that $|G|$ can be larger according to the value of the p -rank γ of the curve C . Indeed, if $\gamma > 0$, then $|G| \leq \frac{2p}{p-1}g$, whereas for $\gamma = 0$, $|G| \leq \max\{g, \frac{4p}{(p-1)^2}g^2\}$, knowing that the quadratic upper bound $\frac{4p}{(p-1)^2}g^2$ can really be attained. Following Nakajima's work, Lehr and Matignon explore the "big actions", that is to say the pairs (C, G) where G is a p -subgroup of $\text{Aut}_k(C)$ such that $\frac{|G|}{g} > \frac{2p}{p-1}$ (see [LM05]). In this case, the ramification locus of the cover $\pi : C \rightarrow C/G$ is located at one point of C , say ∞ . In [MR08], we display necessary conditions on G_2 , the second ramification group of G at ∞ in lower notation, for (C, G) to be a big action. In particular, we show that G_2 coincides with the derived subgroup G' of G .

Motivation and purpose. The aim of this paper is to pursue the classification of big actions as initiated in [LM05]. Indeed, when searching for a classification of big actions, it naturally occurs that the quotient $\frac{|G|}{g^2}$ has a "sieve" effect. Lehr and Matignon first prove that the big actions such that $\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}$ correspond to the p -cyclic étale covers of the affine line parametrized by an Artin-Schreier equation: $W^p - W = f(X) := X S(X) + cX \in k[X]$, where $S(X)$ runs over the additive polynomials of $k[X]$. In [MR08], we show that the big actions satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$ correspond to the étale covers of the affine line with Galois group $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \leq 3$. This motivated the study of big actions with a p -elementary abelian G' , say $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$, which is the main topic of [Ro08a] where we generalize the structure theorem obtained in the p -cyclic case. Namely, we prove that when $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$ with $n \geq 1$, then the function field of the curve is parametrized by n Artin-Schreier equations: $W_i^p - W_i = f_i(X) \in k[X]$ where each function f_i can be written as a linear combination over k of products of at most $i+1$ additive polynomials. In this paper, we display the parametrization of the functions f_i 's in the case of big actions satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$. In what follows, this condition is called condition (*).

Outline of the paper. The paper falls into two main parts. The first one is focused on finiteness results for big actions (C, G) satisfying $\frac{|G|}{g^2} \geq M$ for a given positive real $M > 0$, called big actions satisfying \mathcal{G}_M , whereas the second part is dedicated to the classification of such big actions when $M = \frac{4}{(p^2-1)^2}$. More precisely, we prove in section 4 that, for a given $M > 0$, the order of G' only takes a finite number of values for (C, G) a big action satisfying \mathcal{G}_M . When exploring similar finiteness results for g and $|G|$, we are lead to a purely group-theoretic discussion around the inclusion

$\text{Fratt}(G') \subset [G', G]$, where $\text{Fratt}(G')$ means the Frattini subgroup of G' and $[G', G]$ denotes the commutator subgroup of G' and G (cf. section 4). When the inclusion is strict, $|G|$ and g also take a finite number of values for (C, G) satisfying \mathcal{G}_M . This is no more true when $\text{Fratt}(G') = [G', G]$. In this case, we can only conclude that, for $p > 2$, the quotient $\frac{|G|}{g^2}$ takes a finite number of values for (C, G) satisfying \mathcal{G}_M with an abelian G' . Note that we do not know yet examples of big actions with a non-abelian G' . Another central question is the link between the subgroups G of $\text{Aut}_k(C)$ such that (C, G) is a big action and a p -Sylow subgroup of $\text{Aut}_k(C)$ containing G (section 3). Among other things, we prove that they have the same derived subgroup. This, together with the fact that the order of G' takes a finite number of values for big actions satisfying \mathcal{G}_M , implies, on the one hand, that the order of G' is a key criterion to classify big actions and, on the other hand, that we can concentrate on p -Sylow subgroups of A . In section 5, we eventually display the classification and the parametrization of big actions (C, G) under condition $(*)$ according to the order of G' . Pursuing the preceding discussion, we have to distinguish the cases $[G', G] = \text{Fratt}(G') (= \{e\})$ and $[G', G] \supsetneq \text{Fratt}(G') (= \{e\})$.

Notation and preliminary remarks. Let k be an algebraically closed field of characteristic $p > 0$. We denote by F the Frobenius endomorphism for a k -algebra. Then, \wp means the Frobenius operator minus identity. We denote by $k\{F\}$ the k -subspace of $k[X]$ generated by the polynomials $F^i(X)$, with $i \in \mathbb{N}$. It is a ring under the composition. Furthermore, for all α in k , $F\alpha = \alpha^p F$. The elements of $k\{F\}$ are the additive polynomials, i.e. the polynomials $P(X)$ of $k[X]$ such that for all α and β in k , $P(\alpha + \beta) = P(\alpha) + P(\beta)$. Moreover, a separable polynomial is additive if and only if the set of its roots is a subgroup of k (see [Go96] chap. 1).

Let $f(X)$ be a polynomial of $k[X]$. Then, there is a unique polynomial $\text{red}(f)(X)$ in $k[X]$, called the reduced representative of f , which is p -power free, i.e. $\text{red}(f)(X) \in \bigoplus_{(i,p)=1} kX^i$, and such that $\text{red}(f)(X) = f(X) \bmod \wp(k[X])$. We say that the polynomial f is reduced mod $\wp(k[X])$ if and only if it coincides with its reduced representative $\text{red}(f)$. The equation $W^p - W = f(X)$ defines a p -cyclic étale cover of the affine line that we denote by C_f . Conversely, any p -cyclic étale cover of the affine line $\text{Spec } k[X]$ corresponds to a curve C_f where f is a polynomial of $k[X]$ (see [Mi80] III.4.12, p. 127). By Artin-Schreier theory, the covers C_f and $C_{\text{red}(f)}$ define the same p -cyclic covers of the affine line. The curve C_f is irreducible if and only if $\text{red}(f) \neq 0$.

Throughout the text, C denotes a connected nonsingular projective curve over k , with genus $g \geq 2$. We denote by $A := \text{Aut}_k C$ the k -automorphism group of the curve C and by $S(A)_p$ any p -Sylow subgroup of A . For any point $P \in C$ and any $i \geq -1$, we denote by $A_{P,i}$ the i -th ramification group of A at P in lower notation, namely

$$A_{P,i} := \{\sigma \in A, v_P(\sigma(t_P) - t_P) \geq i + 1\}$$

where t_P denotes a uniformizing parameter at P and v_P means the order function at P .

2 The setting: generalities about big actions.

Definition 2.1. Let C be a connected nonsingular projective curve over k , with genus $g \geq 2$. Let G be a subgroup of A . We say that the pair (C, G) is a big action if G is a finite p -group such that

$$\frac{|G|}{g} > \frac{2p}{p-1}$$

To precise the background of this work, we first recall basic properties of big actions established in [LM05] and [MR08].

Recall 2.2. Assume that (C, G) is a big action. Then, there is a point of C (say ∞) such that G is the wild inertia subgroup of G at ∞ : G_1 . Moreover, the quotient C/G is isomorphic to the projective line \mathbb{P}_k^1 and the ramification locus (respectively branch locus) of the cover $\pi : C \rightarrow C/G$ is the point ∞ (respectively $\pi(\infty)$). For all $i \geq 0$, we denote by G_i the i -th lower ramification group of G at ∞ :

$$G_i := \{\sigma \in G, v_\infty(\sigma(t_\infty) - t_\infty) \geq i + 1\}$$

where t_∞ denotes a uniformizing parameter at ∞ and v_∞ means the order function at ∞ .

1. Then, G_2 is non trivial and it is strictly included in G_1 .
2. The quotient curve C/G_2 is isomorphic to the projective line \mathbb{P}_k^1 .

3. The quotient group G/G_2 acts as a group of translations of the affine line $C/G_2 - \{\infty\} = \text{Spec } k[X]$, through $X \rightarrow X + y$, where y runs over a subgroup V of k . Then, V is an \mathbb{F}_p -subvector space of k . We denote by v its dimension. This gives the following exact sequence:

$$0 \longrightarrow G_2 \longrightarrow G = G_1 \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0$$

where

$$\pi : \begin{cases} G \rightarrow V \\ g \rightarrow g(X) - X \end{cases}$$

Recall 2.3. ([MR08] Thm. 2.6.4). Let (C, G) be a big action. Then,

$$G_2 = G' = \text{Fratt}(G)$$

where G' means the commutator subgroup of G and $\text{Fratt}(G) = G'G^p$ the Frattini subgroup of G .

To conclude this first section, we introduce new definitions used in our future classification.

Definition 2.4. Let C be a connected nonsingular projective curve over k , with genus $g \geq 2$. Let G be a subgroup of A . Let $M > 0$ be a positive real. We say that:

1. G satisfies $\mathcal{G}(C)$ (or (C, G) satisfies \mathcal{G}) if (C, G) is a big action.
2. G satisfies $\mathcal{G}_M(C)$ (or (C, G) satisfies \mathcal{G}_M) if (C, G) is a big action with $\frac{|G|}{g^2} \geq M$.
3. If (C, G) satisfies \mathcal{G}_M with $M = \frac{4}{(p^2-1)^2}$, we say that (C, G) satisfies condition $(*)$.

Remark 2.5. There exists big actions (C, G) satisfying \mathcal{G}_M if and only if $M \leq \frac{4p}{(p-1)^2}$ (see [St73]).

3 A study on p -Sylow subgroups of $\text{Aut}_k(C)$ inducing big actions.

In this section, we more specifically concentrate on the p -Sylow subgroup(s) of A satisfying $\mathcal{G}(C)$ (resp. $\mathcal{G}_M(C)$).

Remark 3.1. Let C be a connected nonsingular projective curve over k , with genus $g \geq 2$. Assume that there exists a subgroup $G \subset A$ satisfying $\mathcal{G}(C)$.

1. Then, every p -Sylow subgroup of A satisfies $\mathcal{G}(C)$.
2. Moreover, A has a unique p -Sylow subgroup except in the three following cases (cf. [Han92] and [GK07]):

(a) The Hermitian curve

$$C_H : W^q + W = X^{1+q}$$

with $p \geq 2$, $q = p^s$, $s \geq 1$. Then, $g = \frac{1}{2}(q^2 - q)$ and $A \simeq \text{PSU}(3, q)$ or $A \simeq \text{PGU}(3, q)$. It follows that $|A| = q^3(q^2 - 1)(q^3 + 1)$, so $\frac{|S(A)_p|}{g} = \frac{2q^2}{q-1} > \frac{2p}{p-1}$ and $\frac{|S(A)_p|}{g^2} = \frac{4q}{(q-1)^2}$, where $S(A)_p$ denotes any p -Sylow subgroup of A . Thus, $(C_H, S(A)_p)$ is a big action with $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^s$. It satisfies condition $(*)$ if and only if $1 \leq s \leq 3$.

(b) The Deligne-Lusztig curve arising from the Suzuki group

$$C_S : W^q + W = X^{q_0}(X^q + X)$$

with $p = 2$, $q_0 = 2^s$, $s \geq 1$ and $q = 2^{2s+1}$. In this case, $g = q_0(q-1)$ and $A \simeq \text{Sz}(q)$ the Suzuki group. It follows that $|A| = q^2(q-1)(q^2+1)$, so $\frac{|S(A)_p|}{g} = \frac{q^2}{q_0(q-1)} > \frac{2p}{p-1}$ and $\frac{|S(A)_p|}{g^2} = \frac{q^2}{q_0^2(q-1)^2} < \frac{4}{(p^2-1)^2}$, for all $s \geq 1$. Thus, $(C_S, S(A)_p)$ is a big action with $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^{2s+1}$ but it never satisfies condition $(*)$.

(c) *The Deligne-Lusztig curve arising from the Ree group*

$$C_R : \quad W_1^q - W_1 = X^{q_0} (X^q + X) \quad \text{and} \quad W_2^q - W_2 = X^{2q_0} (X^q + X)$$

with $p = 3$, $q_0 = 3^s$, $s \geq 1$ and $q = 3^{2s+1}$. Then, $g = \frac{3}{2} q_0 (q-1)(q+q_0+1)$ and $A \simeq \text{Ree}(q)$ the Ree group. It follows that $|A| = q^3 (q-1)(q^3+1)$, so $\frac{|S(A)_p|}{g} = \frac{2q^3}{3q_0(q-1)(q+q_0+1)} > \frac{2p}{p-1}$ and $\frac{|S(A)_p|}{g^2} = \frac{4q^3}{9q_0^2(q-1)^2(q+q_0+1)^2} < \frac{4}{(p^2-1)^2}$ for all $s \geq 1$. Thus, $(C_R, S(A)_p)$ is a big action with $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^{2(2s+1)}$ but it never satisfies condition (*).

In each of these three cases, the group A is simple, so it has more than one p -Sylow subgroups.

Now, fix C a connected nonsingular projective curve over k , with genus $g \geq 2$. We highlight the link between the groups G satisfying $\mathcal{G}(C)$ (resp. $\mathcal{G}_M(C)$) and the p -Sylow subgroup(s) of A .

Proposition 3.2. *Let C be a connected nonsingular projective curve over k , with genus $g \geq 2$.*

1. *Let G satisfy $\mathcal{G}(C)$.*

(a) *Then, there exists a point of C , say ∞ , such that G is included in $A_{\infty,1}$. For all $i \geq 0$, we denote by G_i the i -th ramification group of G at ∞ in lower notation. Then, $A_{\infty,1}$ satisfies $\mathcal{G}(C)$ and $A_{\infty,2} = G_2$, i.e. $(A_{\infty,1})' = G'$. Thus, we obtain the following diagram:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{\infty,2} & \longrightarrow & A_{\infty,1} & \xrightarrow{\pi} & W \subset k \longrightarrow 0 \\ & & \parallel & & \cup & & \cup \\ 0 & \longrightarrow & G_2 & \longrightarrow & G = G_1 & \xrightarrow{\pi} & V \longrightarrow 0 \end{array}$$

In particular, $G = \pi^{-1}(V)$ where V is an \mathbb{F}_p -subvector space of W .

(b) *$A_{\infty,1}$ is a p -Sylow subgroup of A . Moreover, except in the three special cases mentionned in Remark 3.1, $A_{\infty,1}$ is the unique p -Sylow subgroup of A .*

(c) *Let M be a positive real such that G satisfies $\mathcal{G}_M(C)$. Then, $A_{\infty,1}$ also satisfies $\mathcal{G}_M(C)$.*

2. *Conversely, let ∞ be a point of the curve C such that $A_{\infty,1}$ satisfies $\mathcal{G}(C)$. Consider V an \mathbb{F}_p -vector space of W , defined as above, and put $G := \pi^{-1}(V)$.*

(a) *Then, the group G satisfies $\mathcal{G}(C)$ if and only if*

$$|W| \geq |V| > \frac{2p}{p-1} \frac{g}{|A_{\infty,2}|}$$

(b) *Let M be a positive real such that $A_{\infty,1}$ satisfies $\mathcal{G}_M(C)$. Then, G satisfies $\mathcal{G}_M(C)$ if and only if*

$$|W| \geq |V| \geq M \frac{g^2}{|A_{\infty,2}|}$$

Proof: The first assertion (1.a) derives from [LM05] (Prop 8.5) and [MR08] (Cor. 2.10). The second point (1.b) comes from [MR08] (Rem 2.11) together with Remark 3.1. The other claims are obtained via calculation. \square

Remark 3.3. *Except in the three special cases mentionned in Remark 3.1, the point ∞ of C defined in Proposition 3.2 is uniquely determined. In particular, except for the three special cases, if P is a point of C such that $A_{P,1}$ satisfies $\mathcal{G}(C)$, then $P = \infty$.*

As a conclusion, if G satisfies $\mathcal{G}(C)$ (resp. $\mathcal{G}_M(C)$) and if $A_{\infty,1}$ is a (actually "the", in most cases) p -Sylow subgroup of A containing G , then $A_{\infty,1}$ also satisfies $\mathcal{G}(C)$ (resp. $\mathcal{G}_M(C)$) and has the same derived subgroup. So, in our attempt to classify the big actions (C, G) satisfying \mathcal{G}_M , this leads us to focus on the derived subgroup G' of G .

4 Finiteness results for big actions satisfying \mathcal{G}_M .

4.1 An upper bound on $|G'|$.

Lemma 4.1. *Let $M > 0$ be a positive real such that (C, G) is a big action satisfying \mathcal{G}_M . Then, the order of G' is bounded as follows:*

$$p \leq |G'| \leq \frac{4p}{(p-1)^2} \frac{2+M+2\sqrt{1+M}}{M^2}$$

Thus, $|G'|$ only takes a finite number of values for (C, G) a big action satisfying \mathcal{G}_M .

Proof: We first recall that $G' = G_2$ is a non-trivial p -group (see e.g. [LM05] Prop. 8.5). Now, let $i_0 \geq 2$ be the integer such that the lower ramification filtration of G at ∞ reads:

$$G = G_0 = G_1 \supsetneq G_2 = \cdots = G_{i_0} \supsetneq G_{i_0+1} = \cdots$$

Put $|G_2/G_{i_0+1}| = p^m$, with $m \geq 1$, and $\mathcal{B}_m := \frac{4}{M} \frac{|G_2/G_{i_0+1}|}{(|G_2/G_{i_0+1}|-1)^2} = \frac{4}{M} \frac{p^m}{(p^m-1)^2}$. By [LM05] (Thm. 8.6), $M \leq \frac{|G|}{g^2}$ implies $1 < |G_2| \leq \frac{4}{M} \frac{|G_2/G_{i_0+1}|^2}{(|G_2/G_{i_0+1}|-1)^2} = p^m \mathcal{B}_m$. From $|G_2| = p^m |G_{i_0+1}|$, we infer $1 \leq |G_{i_0+1}| \leq \mathcal{B}_m$. Since $(\mathcal{B}_m)_{m \geq 1}$ is a decreasing sequence which tends to 0 as m grows large, we conclude that m is bounded. More precisely, $m < m_0$ where m_0 is the smallest integer such that $\mathcal{B}_{m_0} < 1$. As $M \leq \frac{4p}{(p-1)^2} \leq 8$ (see Remark 2.5), computation shows that $\mathcal{B}_m < 1 \Leftrightarrow p^m > \phi(M) := \frac{2+M+2\sqrt{1+M}}{M}$. As $(\mathcal{B}_m)_{m \geq 1}$ is decreasing,

$$|G_2| \leq p^m \mathcal{B}_m \leq \phi_1(M) \mathcal{B}_1 = \frac{\phi(M)}{M} \frac{4p}{(p-1)^2}$$

The claim follows. \square

We deduce that, for big actions (C, G) satisfying \mathcal{G}_M , an upper bound on $|V|$ induces an upper bound on the genus g of C .

Corollary 4.2. *Let $M > 0$ be a positive real such that (C, G) is a big action satisfying \mathcal{G}_M . Then,*

$$g < \frac{|G'| |V|}{2p} \leq \frac{2}{p-1} \frac{2+M+2\sqrt{1+M}}{M^2} |V|$$

This raises the following question. Let (C, G) be a big action satisfying \mathcal{G}_M ; in which cases is $|V|$ (and then g) bounded from above? In other words, in which cases, does the quotient $\frac{|G|}{g}$ take a finite number of values when (C, G) satisfy \mathcal{G}_M ? We begin with preliminary results on big actions leading to a purely group-theoretic discussion leading to compare the Frattini subgroup of G' with the commutator subgroup of G' and G .

4.2 Preliminaries to a group-theoretic discussion.

Lemma 4.3. *Let (C, G) be a big action. If $G' \subset Z(G)$, then $G' (= G_2)$ is p -elementary abelian, say $G' \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. In this case, the function field $L = k(C)$ is parametrized by n equations:*

$$\forall i \in \{1, \dots, n\}, \quad W_i^p - W_i = f_i(X) = X S_i(X) + c_i X \in k[X]$$

where S_i is an additive polynomial of $k[X]$ with degree $s_i \geq 1$ in F and $s_1 \leq s_2 \leq \dots \leq s_n$. Moreover, $V \subset \cap_{1 \leq i \leq n} Z(\text{Ad}_{f_i})$ where Ad_{f_i} denotes the palindromic polynomial related to f_i as defined in [Ro08a] (Prop. 2.13)

Proof: The hypothesis first requires $G' = G_2$ to be abelian. Now, assume that G_2 has exponent strictly greater than p . Then, there exists a surjective map $\phi : G_2 \rightarrow \mathbb{Z}/p^2\mathbb{Z}$. So $H := \text{Ker} \phi \subsetneq G_2 \subset Z(G)$ is a normal subgroup of G . It follows from [MR08] (Lemma 2.4) that the pair $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 \simeq \mathbb{Z}/p^2\mathbb{Z}$. This contradicts [MR08] (Thm. 5.1). The last part of the lemma comes from [Ro08a] (Prop. 2.13). \square

Corollary 4.4. *Let (C, G) be a big action. Let $H := [G', G]$ be the commutator subgroup of G' and G .*

1. *Then, H is trivial if and only if $G' \subset Z(G)$.*
2. *The group H is strictly included in G' .*
3. *The pair $(C/H, G/H)$ is a big action. Moreover, its second ramification group $(G/H)_2 = (G/H)' = G_2/H \subset Z(G/H)$ is p -elementary abelian.*

Proof:

1. The first assertion is clear.
2. As G' is normal in G , then $H \subset G'$. Assume that $G' = H$. Then, the lower central series of G is stationary, which contradicts the fact that the p -group G is nilpotent (see e.g. [Su86] Chap.4). So $H \subsetneq G'$.

3. As $H \subsetneq G' = G_2$ is normal in G , it follows from [MR08] (Lemma 2.4 and Thm. 2.6) that the pair $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 = G_2/H$. From $H = [G_2 : G]$, we gather that $G_2/H \subset Z(G/H)$. Therefore, we deduce from Lemma 4.3 that $(G/H)_2$ is p -elementary abelian. \square

Corollary 4.5. *Let (C, G) be a big action. Let $F := \text{Fratt}(G')$ be the Frattini subgroup of G' .*

1. *Then, F is trivial if and only if G' is an elementary abelian p -group.*
2. *We have the following inclusions: $F \subset [G', G] \subsetneq G'$.*
3. *The pair $(C/F, G/F)$ is a big action. Moreover, its second ramification group $(G/F)_2 = (G/F)' = G_2/F$ is p -elementary abelian.*
4. *Let M be a positive real. If (C, G) satisfies \mathcal{G}_M , then $(C/F, G/F)$ also satisfies \mathcal{G}_M .*

Proof:

1. As G' is a p -group, $F = (G')'(G')^p$, where $(G')'$ means the derived subgroup of G' and $(G')^p$ the subgroup generated by the p powers of elements of G' (cf. [LGMK02] Prop. 1.2.4). This proves that if G' is p -elementary abelian, then F is trivial. The converse derives from the fact that G'/F is p -elementary abelian (cf. [LGMK02] Prop. 1.2.4).
2. Using Corollary 4.4, the only inclusion that remains to show is $F \subset [G', G]$. As $G'/[G', G]$ is abelian, $(G')' \subset [G', G]$. As $G'/[G', G]$ has exponent p , $(G')^p \subset [G', G]$. The claim follows.
3. Since $F \subsetneq G' = G_2$ is normal in G , we deduce from [MR08] (Lemma 2.4) that the pair $(C/F, G/F)$ is a big action with second ramification group: $(G/F)_2 = G_2/F = (G/F)'$. Furthermore, as G_2 is a p -group, G_2/F is an elementary abelian p -group (see above).
4. This derives from [LM05] (Prop. 8.5 (ii)). \square

This leads us to discuss according to whether $\text{Fratt}(G') \subsetneq [G', G]$ or $\text{Fratt}(G') = [G', G]$.

4.3 Case: $\text{Fratt}(G') \subsetneq [G', G]$

We start with the special case $\{e\} = \text{Fratt}(G') \subsetneq [G', G]$, i.e. G' is p -elementary abelian and $G' \not\subset Z(G)$.

Proposition 4.6. *Let $M > 0$ be a positive real such that (C, G) is a big action satisfying \mathcal{G}_M . Suppose that $\{e\} = \text{Fratt}(G') \subsetneq [G', G]$. Then, $|V|$ and g are bounded as follows:*

$$|V| \leq \frac{4}{M} \frac{|G_2|}{(p-1)^2} \leq \frac{16p}{(p-1)^4} \frac{2+M+2\sqrt{1+M}}{M^3} \quad (1)$$

and

$$\frac{p-1}{2} |V| \leq g < \frac{32p}{(p-1)^5} \frac{(2+M+2\sqrt{1+M})^2}{M^5} \quad (2)$$

Thus, under these conditions, g , $|V|$ and so the quotient $\frac{|G|}{g}$ only take a finite number of values.

Proof: Write $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. As $G_2 \not\subset Z(G)$, [Ro08a] (Prop. 2.13) ensures the existence of a smaller integer $j_0 \geq 1$ such that $f_{j_0+1}(X)$ cannot be written as $cX + XS(X)$, with S in $k\{F\}$. If $j_0 \geq 2$, it follows that, for all y in V , the coefficients of the matrix $L(y)$ satisfy $\ell_{j,i}(y) = 0$ for all $2 \leq i \leq j_0$ and $1 \leq j \leq i-1$. Moreover, the matricial multiplication proves that, for all i in $\{1, \dots, j_0\}$, the functions ℓ_{i,j_0+1} are nonzero linear forms from V to \mathbb{F}_p . Put $\mathcal{W} := \bigcap_{1 \leq i \leq j_0} \ker \ell_{i,j_0+1}$. Let $C_{f_{j_0+1}}$ be the curve parametrized by $W^p - W = f_{j_0+1}(X)$. It defines an étale cover of the affine line with group $\Gamma_0 \simeq \mathbb{Z}/p\mathbb{Z}$. Since, for all y in \mathcal{W} , $f_{i_0+1}(X+y) = f_{i_0+1}(X) \pmod{\wp(k[X])}$, the group of translations of the affine line: $\{X \rightarrow X+y, y \in \mathcal{W}\}$ can be extended to a p -group of automorphisms of the curve $C_{f_{j_0+1}}$, say Γ , with the following exact sequence:

$$0 \longrightarrow \Gamma_0 \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow \Gamma \longrightarrow \mathcal{W} \longrightarrow 0$$

The pair $(C_{f_{j_0+1}}, \Gamma)$ is not a big action. Otherwise, its second ramification group would be p -cyclic, which contradicts the form of the function $f_{j_0+1}(X)$, as compared with [MR08] (Prop. 2.5). Thus,

$\frac{|V|}{gC_{f_{j_0+1}}} = \frac{2p}{p-1} \frac{|W|}{(m_{j_0+1}-1)} \leq \frac{2p}{p-1}$. The inequality $\frac{|V|}{p^{j_0}} \leq |W| \leq (m_{j_0+1}-1)$ combined with the formula given in [Ro08a] (Cor. 2.7) yields a lower bound on the genus, namely:

$$g = \frac{p-1}{2} \sum_{i=1}^n p^{i-1} (m_i - 1) \geq \frac{p-1}{2} p^{j_0} (m_{j_0+1} - 1) \geq \frac{p-1}{2} |V|.$$

It follows that $M \leq \frac{|G|}{g^2} = \frac{|G_2||V|}{g^2} \leq \frac{4|G_2|}{(p-1)^2|V|}$. Using Lemma 4.1, we gather inequality (1). Inequality (2) then derives from Corollary 4.2. \square

The following corollary generalizes the finiteness result of Proposition 4.6 to all big actions satisfying \mathcal{G}_M such that $\text{Fratt}(G') \subsetneq [G', G]$.

Corollary 4.7. *Let $M > 0$ be a positive real such that (C, G) is a big action satisfying \mathcal{G}_M . Suppose that $\text{Fratt}(G') \subsetneq [G', G]$. Then, $|V|$ and g are bounded as in Proposition 4.6. So the quotients $\frac{|G|}{g}$ and $\frac{|G|}{g^2}$ only take a finite number of values.*

Proof: Put $F := \text{Fratt}(G')$. Corollary 4.5 asserts that the pair $(C/F, G/F)$ is a big action satisfying \mathcal{G}_M whose second ramification group: $(G/F)_2 = G_2/F$ is p -elementary abelian. From $F \subsetneq [G_2 : G]$, we gather $\{e\} \subsetneq [G_2/F : G/F]$, which implies $(G/F)_2 = (G/F)' \not\subset Z(G/F)$. We deduce that $|V|$ is bounded from above as in Proposition 4.6. The claim follows. \square

4.4 Case: $\text{Fratt}(G') = [G', G]$

It remains to investigate the case where $\text{Fratt}(G') = [G', G]$. In particular, this equality is satisfied when G' is included in the center of G and so is p -elementary abelian (cf. Lemma 3.3), i.e. $\{e\} = \text{Fratt}(G') = [G', G]$. The finiteness result on g obtained in the preceding section is no more true in this case, as illustrated by the remark below.

Remark 4.8. *For any integer $s \geq 1$, Proposition 2.5 in [MR08] exhibits an example of big actions (C, G) with $C : W^p - W = X S(X)$ where S is an additive polynomial of $k[X]$ with degree p^s . In this case, $g = \frac{p-1}{2} p^s$, $V = Z(\text{Ad}_f) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s}$ and $G' = G_2 \simeq \mathbb{Z}/p\mathbb{Z} \subset Z(G)$. It follows that $\frac{|G|}{g^2} = \frac{4p}{(p-1)^2}$. So, for all $M \leq \frac{4p}{(p-1)^2}$, (C, G) satisfies \mathcal{G}_M , with $\{e\} = \text{Fratt}(G') = [G', G]$, whereas $g = \frac{p-1}{2} p^s$ grows arbitrary large with s .*

Therefore, in this case, neither g nor $|V|$ are bounded. Nevertheless, the following section shows that, under these conditions, the quotient $\frac{|G|}{g^2}$ take a finite number of values.

4.4.1 Case: $\text{Fratt}(G') = [G', G] = \{e\}$.

Proposition 4.9. *Let $M > 0$ be a positive real such that (C, G) is a big action satisfying \mathcal{G}_M . Assume that $[G', G] = \text{Fratt}(G') = \{e\}$. Let s_1 be the integer in Lemma 4.3. Then, $\frac{p^{2s_1}}{g^2}$ and $\frac{|V|}{p^{2s_1}}$ are bounded as follows:*

$$\frac{p^{2s_1}}{g^2} \leq \frac{(p-1)^2}{4p} \frac{M^3}{2 + M + 2\sqrt{1+M}} \quad (3)$$

and

$$1 \leq \frac{|V|}{p^{2s_1}} \leq \frac{(p-1)^4}{16p} \frac{M^3}{2 + M + 2\sqrt{1+M}} \quad (4)$$

Thus, the quotient $\frac{|G|}{g^2}$ takes a finite number of values.

Proof: Write $G' = G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n$, with $n \geq 1$. Lemma 4.1 first implies that p^n can only take a finite number of values. Moreover, as recalled in Lemma 4.3, $V \subset \bigcap_{i=1}^n Z(\text{Ad}_{f_i})$ and $|G| = |G_2||V| \leq p^{n+2s_1}$. We compute the genus by means of [Ro08a] (Cor. 2.7):

$$g = \frac{p-1}{2} \sum_{i=1}^n p^{i-1} (m_i - 1) = \frac{p-1}{2} p^{s_1} \left(\sum_{i=1}^n p^{i-1} p^{s_i - s_1} \right)$$

It follows that: $0 < M \leq \frac{|G|}{g^2} \leq \frac{4p^n}{(p-1)^2 (\sum_{i=1}^n p^{i-1} p^{s_i - s_1})^2}$. This implies $(\sum_{i=1}^n p^{i-1} p^{s_i - s_1})^2 \leq \frac{4p^n}{M(p-1)^2}$. As p^n is bounded from above, the set $\{s_i - s_1, i \in [1, n]\} \subset \mathbb{N}$ is also bounded, and then finite. More precisely, we gather that

$$\frac{g^2}{p^{2s_1}} = \frac{(p-1)^2}{4} \left(\sum_{i=1}^n p^{i-1} p^{s_i - s_1} \right)^2 \leq \frac{p^n}{M}$$

Combined with Lemma 4.1, this gives inequality (3). Besides, from $M \leq \frac{|G|}{g^2} = \frac{|V|p}{g^2}$, we infer that $\frac{1}{|V|} \leq \frac{p^n}{Mg^2}$, which involves:

$$1 \leq \frac{p^{2s_1}}{|V|} \leq \frac{p^{2s_1} p^n}{Mg^2} = \frac{4p^n}{M(p-1)^2 (\sum_{i=1}^n p^{i-1} p^{s_i - s_1})^2} \leq \frac{4p^n}{M(p-1)^2}$$

This, together with Lemma 4.1, yields inequality (4). In particular, the set $\{\frac{p^{2s_1}}{|V|}\} \subset \mathbb{N}$ is bounded, and then finite, as well as the set $\{\frac{|V|}{p^{2s_1}}\}$. Therefore, the quotient $\frac{|G|}{g^2} = p^n \frac{|V|}{p^{2s_1}} \frac{p^{2s_1}}{g^2}$ can only take a finite number of values. \square

The last remaining case is $\text{Fratt}(G') = [G', G] \neq \{e\}$.

4.4.2 Case: $\text{Fratt}(G') = [G', G] \neq \{e\}$.

As shown below, this case can only occur for $G' (= G_2)$ non abelian. Note that we do not know yet examples of big actions with a non abelian $G' (= G_2)$.

Theorem 4.10. *Assume that $p > 2$. Let (C, G) be a big action with $\text{Fratt}(G') = [G', G] \neq \{e\}$. Then, $G' (= G_2)$ is non abelian.*

We deduce the following

Corollary 4.11. *Assume that $p > 2$. Let $M > 0$ be a positive real. Let (C, G) be a big action satisfying \mathcal{G}_M with G' abelian. Then, $\frac{|G|}{g^2}$ only takes a finite number of values.*

Remark 4.12. *Theorem 4.10 is no more true for $p = 2$. A counterexample is given by [MR08] (Prop. 6.9) applied with $p = 2$. Indeed, when keeping the notations of [MR08] (Prop. 6.9), take $q = p^e$ with $p = 2$, $e = 2s - 1$ and $s \geq 2$. Put $K = \mathbb{F}_q(X)$. Let $L := \mathbb{F}_q(X, W_1, V_1, W_2)$ be the extension of K parametrized by*

$$W_1^{2^{2s-1}} - W_1 = X^{2^{s-1}} (X^{2^{2s-1}} - X) \quad V_1^{2^{2s-1}} - V_1 = X^{2^{s-2}} (X^{2^{2s-1}} - X)$$

$$[W_1, W_2]^2 - [W_1, W_2] = [X^{1+2^s}, 0] - [X^{1+2^{s-1}}, 0]$$

Let G be the p -group of \mathbb{F}_q -automorphisms of L constructed as in [MR08] (Prop. 6.9.3). Then, the formula established for g_L in [MR08] (Prop. 6.9.4) shows that the pair (C, G) is a big action as soon as $s \geq 4$. In this case, $G' = G_2 \simeq \mathbb{Z}/2^2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{6s-4}$ (cf. [MR08] Prop. 6.7.2). As the functions $X^{2^{s-1}} (X^{2^{2s-1}} - X)$ and $X^{2^{s-2}} (X^{2^{2s-1}} - X)$ are products of two additive polynomials, it follows from next proof (cf. point 6) that $[G', G] = \text{Fratt}(G') \neq \{e\}$.

Proof of Theorem 4.10:

1. *Preliminary remarks: the link with Theorem 5.1 in [MR08].*

- (a) One first remarks that Theorem 4.10 implies Theorem 5.1 in [MR08]. The latter states that there is no big action (C, G) with G_2 cyclic of exponent strictly greater than p . Indeed, assume that there exists one. Then, $G' = G_2$ is abelian and $\text{Fratt}(G') = (G')^p \neq \{e\}$. To contradicts Theorem 4.10, it remains to show that $F := \text{Fratt}(G') = [G', G]$. From Corollary 4.5, we infer that $(C/F, G/F)$ is a big action whose second ramification group G_2/F is cyclic of order p . Then, $(G/F)' = (G/F)_2 = G_2/F \subset Z(G/F)$ (cf. [MR08] Prop. 2.5 and [Ro08a] Prop. 2.13). It follows that $\text{Fratt}((G/F)') = [(G/F)', G/F] = \{e\}$. As $F \subset G'$, this imposes $F = [G', G]$. Then, Theorem 4.10 contradicts the fact that $G' = G_2$ is abelian.
- (b) The object of Theorem 4.10 is to prove that there exists no big action (C, G) with $G' = G_2$ abelian of exponent strictly greater than p such that $\text{Fratt}(G') = [G', G]$. The proof follows the same canvas as the proof of [MR08] (Thm. 5.1). Nevertheless, to refine the arguments, we use the formalism related to the ring filtration of $k[X]$ linked with the additive polynomials as introduced in [Ro08a] (cf. section 3). More precisely, we recall that, for any $t \geq 1$, we define Σ_t as the k -subvector space of $k[X]$ generated by 1 and the products of at most t additive polynomials of $k[X]$ (cf. [Ro08a] Def. 3.1). In what follows, we assume that there exists a big action (C, G) with $G' = G_2$ abelian of exponent strictly greater than p such that $\text{Fratt}(G') = [G', G]$.

2. *One can suppose that $G' = G_2 \simeq \mathbb{Z}/p^2\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^r$, with $r \geq 1$.*
 Indeed, write $G'/(G')^{p^2} \simeq (\mathbb{Z}/p^2\mathbb{Z})^a \times (\mathbb{Z}/p\mathbb{Z})^b$. By assumption, $a \geq 1$. Using [Su82] (Chap.2, Thm. 19), one can find an index p -subgroup of $(G')^p$, normal in G , such that $(G')^{p^2} \subset H \subsetneq (G')^p \subsetneq G' = G_2$. Then, we infer from [MR08] (Lemma 2.4) that $(C/H, G/H)$ is a big action with second ramification group $(G/H)' = (G/H)_2 = G_2/H \simeq (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^{a+b-1}$. Furthermore, as G' is abelian, $\text{Fratt}(G') = (G')^p$ (resp. $\text{Fratt}((G/H)') = ((G/H)')^p$). From $H \subset (G')^p$ with H normal in G and $\text{Fratt}(G') = [G', G]$, we gather that $\text{Fratt}((G/H)') = (G')^p/H = \text{Fratt}(G')/H = [(G/H)', G/H]$.

3. *Notation.*

In what follows, we denote by $L := k(C)$ the function field of C and by $k(X) := L^{G_2}$ the subfield of L fixed by G_2 . Following Artin-Schreier-Witt theory as already used in [MR08] (proof of Thm. 5.1, point 2), we introduce the $W_2(\mathbb{F}_p)$ -module

$$A := \frac{\wp(W_2(L)) \cap W_2(k[X])}{\wp(W_2(k[X]))}$$

where $W_2(L)$ means the ring of Witt vectors of length 2 with coordinates in L and $\wp = F - id$. One can prove that A is isomorphic to the dual of G_2 with respect to the Artin-Schreier-Witt pairing (cf. [Bo83] Chap. IX, ex. 19). Moreover, as a \mathbb{Z} -module, A is generated by the classes mod $\wp(k[X])$ of $(f_0(X), g_0(X))$ and $\{(0, f_i(X))\}_{1 \leq i \leq r}$ in $W_2(k[X])$. In other words, $L = k(X, W_i, V_0)_{0 \leq i \leq r}$ is parametrized by the following system of Artin-Schreier-Witt equations:

$$\wp([W_0, V_0]) = [f_0(X), g_0(X)] \in W_2(k[X])$$

and

$$\forall i \in \{1, \dots, r\}, \quad \wp(W_i) = f_i(X) \in k[X]$$

An exercise left to the reader shows that one can choose $g_0(X)$ and each $f_i(X)$, with $0 \leq i \leq r$, reduced mod $\wp(k[X])$.

4. *We prove that $f_0 \in \Sigma_2$.*

As a \mathbb{Z} -module, pA is generated by the class of $(0, f_0(X))$ in A . By the Artin-Schreier-Witt pairing, pA corresponds to the kernel $G_2[p]$ of the map:

$$\begin{cases} G_2 \rightarrow G_2 \\ g \rightarrow g^p \end{cases}$$

Thus, $G_2[p] \subsetneq G_2$ is a normal subgroup of G . Then, it follows from [MR08] (Lemma 2.4) that the pair $(C/G_2[p], G/G_2[p])$ is a big action parametrized by $W^p - W = f_0(X)$ and with second ramification group $G_2/G_2[p] \simeq \mathbb{Z}/p\mathbb{Z}$. Then, $f_0(X) = X S(X) + cX \in k[X]$ (cf. [MR08] Prop. 2.5), where S is an additive polynomial of $k\{F\}$ with degree $s \geq 1$ in F .

5. *The embedding problem.*

For any $y \in V$, the classes mod $\wp(k[X])$ of $(f_0(X+y), g_0(X+y))$ and $\{(0, f_i(X+y))\}_{1 \leq i \leq r}$ induces a new generating system of A . As in [MR08] (proof of Thm 5.1, point 3), this is expressed by the following equation:

$$\forall y \in V, \quad (f_0(X+y), g_0(X+y)) = (f_0(X), g_0(X) + \sum_{i=0}^r \ell_i(y) f_i(X)) \mod \wp(W_2(k[X])) \quad (5)$$

where, for all i in $\{0, \dots, r\}$, ℓ_i is a linear form from V to \mathbb{F}_p . On the second coordinate, (5) reads:

$$\forall y \in V, \quad \Delta_y(g_0) := g_0(X+y) - g_0(X) = \sum_{i=0}^r \ell_i(y) f_i(X) + c \mod \wp(k[X]) \quad (6)$$

where

$$c = \sum_{i=1}^{p-1} \frac{(-1)^i}{i} y^{p-i} X^{i+p^{s+1}} + \text{lower degree terms in } X \quad (7)$$

For more details on calculation, we refer to [MR08] (proof of Thm 5.1, point 3 and Lemma 5.2).

6. We prove that f_i lies in Σ_2 , for all i in $\{0, \dots, r\}$, if and only if $\text{Fratt}(G') = [G', G]$. Put $F := \text{Fratt}(G')$. We deduce from Corollary 4.5 that $(C/F, G/F)$ is a big action whose second ramification group $(G/F)' = (G/F)_2 = G_2/F$ is p -elementary abelian. The function field of the curve C/F is now parametrized by the Artin-Schreier equations:

$$\forall i \in \{0, \dots, r\}, \quad \wp(W_i) = f_i(X) \in k[X]$$

As $F \subset [G', G]$ (cf. Lemma 4.5),

$$F = [G', G] = [G_2 : G] \Leftrightarrow \{e\} = [G_2/F, G/F] = [(G/F)', G/F] \Leftrightarrow (G/F)' \subset Z(G/F)$$

By [Ro08a] (Prop.2.13), this occurs if and only if for all i in $\{0, \dots, r\}$, $f_i(X) = X S_i(X) + c_i X \in \Sigma_2$.

7. We prove that g_0 does not belong to Σ_p .

We first notice that the right-hand side of (6) does not belong to Σ_{p-1} : indeed, the monomial $X^{p-1+p^{s+1}} \in \Sigma_p - \Sigma_{p-1}$ occurs once in c but not in $\sum_{i=0}^r \ell_i(y) f_i(X)$ which lies in $\Sigma_2 \subset \Sigma_{p-1}$, for $p \geq 3$. Now, assume that $g_0 \in \Sigma_p$. Then, by [Ro08a] (Lemma 3.9), the left-hand side of (6), namely $\Delta_y(g_0)$, lies in Σ_{p-1} , hence a contradiction. Therefore, one can define an integer a such that X^a is the monomial of $g_0(X)$ with highest degree among those that do not belong to Σ_p . Note that since g_0 is reduced mod $\wp(k[X])$, $a \not\equiv 0 \pmod p$.

8. We prove that $a - 1 \geq p - 1 + p^{s+1}$.

We have already seen that the monomial $X^{p-1+p^{s+1}}$ occurs in the right hand side of (6). In the left-hand side of (6), $X^{p-1+p^{s+1}}$ is produced by monomials X^b of g_0 with $b > p - 1 + p^{s+1}$. If $b > a$, $X^b \in \Sigma_p$, so $\Delta_y(X^b) \in \Sigma_{p-1}$, which is not the case of $X^{p-1+p^{s+1}}$. It follows that $X^{p-1+p^{s+1}}$ comes from monomials X^b with $a \geq b > p - 1 + p^{s+1}$. Hence the expected inequality.

9. We prove that p divides $a - 1$.

Assume that p does not divide $a - 1$. In this case, the monomial X^{a-1} is reduced mod $\wp(k[X])$ and (6) reads as follows:

$$\forall y \in V, \quad c_a(g_0) a y X^{a-1} + S_{p-1}(X) + R_{a-2}(X) = c + \sum_{i=0}^r \ell_i(y) f_i(X) \pmod{\wp(k[X])}$$

where $c_a(g_0) \neq 0$ denotes the coefficient of X^a in g_0 , $S_{p-1}(X)$ is a polynomial in Σ_{p-1} produced by monomials X^b of g_0 with $b > a$ and $R_{a-2}(X)$ is a polynomial of $k[X]$ with degree lower than $a - 2$ produced by monomials X^b of g_0 with $b \leq a$. We first notice that X^{a-1} does not occur in $S_{p-1}(X)$. Otherwise, $X^{a-1} \in \Sigma_{p-1}$ and $X^a = X^{a-1} X \in \Sigma_p$, hence a contradiction. Likewise, X^{a-1} does not occur in $\sum_{i=0}^r \ell_i(y) f_i(X) \in \Sigma_2$. Otherwise, $X^a = X^{a-1} X \in \Sigma_3 \subset \Sigma_p$, as $p \geq 3$. It follows that X^{a-1} occurs in c , which requires $a - 1 \leq \deg b = p - 1 + p^{s+1}$. Then, the previous point implies $a - 1 = p - 1 + p^{s+1}$, which contradicts $a \not\equiv 0 \pmod p$.

Thus, p divides $a - 1$. So, we can write $a = 1 + \lambda p^t$, with $t > 0$, λ prime to p and $\lambda \geq 2$ because of the definition of a . We also define $j_0 := a - p^t = 1 + (\lambda - 1)p^t$.

10. We search for the coefficient of the monomial X^{j_0} in the left-hand side of (6).

Since p does not divide j_0 , the monomial X^{j_0} is reduced mod $\wp(k[X])$. In the left-hand side of (6), namely $\Delta_y(g_0) \pmod{\wp(k[X])}$, the monomial X^{j_0} comes from monomials of $g_0(X)$ of the form: X^b , with $b \geq j_0 + 1$. However, as seen above, the monomials X^b with $b > a$ produce in $\Delta_y(g_0)$ elements that belong to Σ_{p-1} , whereas $X^{j_0} \notin \Sigma_{p-1}$. Otherwise, $X^a = X^{j_0} X^{p^t} \in \Sigma_p$, which contradicts the definition of a . So we only have to consider the monomials X^b of $g_0(X)$ with $b \in \{j_0 + 1, \dots, a\}$. Then, the same arguments as those used in [MR08] (proof of Thm. 5.1, point 6) allow to conclude that the coefficient of X^{j_0} in the left-hand side of (6) is $T(y)$ where $T(Y)$ denotes a polynomial of $k[X]$ with degree p^t .

11. We identify with the coefficient of X^{j_0} in the right-hand side of (6) and gather a contradiction.

As mentionned above, the monomial X^{j_0} does not occur in $\sum_{i=0}^r \ell_i(y) f_i(X) \in \Sigma_2 \subset \Sigma_{p-1}$, for $p \geq 3$. Assume that the monomial X^{j_0} appears in c , which implies that $j_0 \leq p - 1 + p^{s+1}$. Using the same arguments as in [MR08] (proof of Thm. 5.1, point 7), we gather that $j_0 = 1 + (\lambda - 1)p^t = 1 + p^{s+1}$. Then, X^{j_0} lies in Σ_2 , which leads to the same contradiction as above. Therefore, the monomial X^{j_0} does not occur in the right-hand side of (6). Then, $T(y) = 0$ for all y in V , which means that $|V| \leq p^t$. Call C_0 the curve whose function field

is parametrized by $\wp([W_0, V_0]) = [f_0(X), g_0(X)]$. The same calculation as in [MR08] (proof of Theorem 5.1, point 7) shows that $g_{C_0} \geq p^{t+1}(p-1)$. Furthermore, $g \geq p^r g_{C_0}$ (see e.g. [LM05] Prop. 8.5, formula (8)). As $|G| = |G_2||V| \leq p^{2+r+t}$, it follows that $\frac{|G|}{g} = \frac{p}{p-1} < \frac{2p}{p-1}$, hence a contradiction. \square

5 Classification of big actions under condition (*).

We now pursue the classification of big actions initiated by Lehr and Matignon who characterize big actions (C, G) satisfying $\frac{|G|}{g^2} \geq \frac{4}{(p-1)^2}$ (cf. [LM05]). In this section, we exhibit a parametrization for big actions (C, G) satisfying condition (*), namely:

$$\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2} \quad (*)$$

As proved in [MR08] (Prop. 4.1 and Prop. 4.2), this condition requires $G' (= G_2)$ to be an elementary abelian p -group with order dividing p^3 . Since G_2 cannot be trivial (cf. [MR08] Prop. 2.2), this leaves three possibilities. This motivates the following

Definition 5.1. *Let (C, G) be abig action. Let $i \geq 1$ be an integer. We say that*

1. (C, G) satisfies \mathcal{G}_* if (C, G) satisfies condition (*)
2. (C, G) satisfies $\mathcal{G}_*^{p^i}$ if (C, G) satisfies \mathcal{G}_* with $G' \simeq (\mathbb{Z}/p\mathbb{Z})^i$.

5.1 Preliminaries: big actions with a p -elementary abelian $G' (= G_2)$.

To start with, we fix the notations and recall some necessary results on big actions with a p -elementary abelian G_2 drawn from [Ro08a].

Recall 5.2. *Let (C, G) be a big action such that $G' (= G_2) \simeq (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 1$. Write the exact sequence:*

$$0 \longrightarrow G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^n \longrightarrow G \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\mathbb{Z})^v \longrightarrow 0$$

1. *We denote by L be the function field of the curve C and by $k(X) := L^{G_2}$ the subfield of L fixed by G_2 . Then, the extension $L/k(X)$ can be parametrized by n Artin-Schreier equations: $W_i^p - W_i = f_i(X) \in k[X]$ with $1 \leq i \leq n$. Following [Ro08a] (Def. 2.3), one can choose an "adapted basis" $\{f_1(X), \dots, f_n(X)\}$ with some specific properties:*

- (a) *For all $i \in \{1, \dots, n\}$, each function f_i is assumed to be reduced mod $\wp(k[X])$*
- (b) *For all $i \in \{1, \dots, n\}$, put $m_i := \deg f_i$. Then, $m_1 \leq m_2 \leq \dots \leq m_n$.*
- (c) *$\forall (\lambda_1, \dots, \lambda_n) \in \mathbb{F}_p^n$ not all zeros,*

$$\deg \left(\sum_{i=1}^n \lambda_i f_i(X) \right) = \max_{i \in \{1, \dots, n\}} \{ \deg \lambda_i f_i(X) \}.$$

In this case, the genus of the curve C is given by the following formula (cf. [Ro08a] Cor. 2.7):

$$g = \frac{p-1}{2} \sum_{i=1}^n p^{i-1} (m_i - 1) \quad (8)$$

2. *Now, consider the \mathbb{F}_p -subvector space of $k[X]$ generated by the classes of $\{f_1(X), \dots, f_n(X)\}$ mod $\wp(k[X])$:*

$$A := \frac{\wp(L) \cap k[X]}{\wp(k[X])}$$

Recall that A is isomorphic to the dual of G_2 with respect to the Artin-Schreier pairing (cf. [Ro08a] section 2.1). As seen in [Ro08a] (section 2.2), V acts on G_2 via conjugation. This induces a representation $\phi: V \rightarrow \text{Aut}(G_2)$. The representation $\rho: V \rightarrow \text{Aut}(A)$, which is dual

with respect to the Artin-Schreier pairing, expresses the action of V on A by translation. More precisely, for all y in V , the automorphism $\rho(y)$ is defined as follows:

$$\rho(y) : \left\{ \begin{array}{l} A \rightarrow A \\ \overline{f(X)} \rightarrow \overline{f(X+y)} \end{array} \right.$$

where $\overline{f(X)}$ means the class in A of $f(X) \in k[X]$. For all y in V , the matrix of the automorphism $\rho(y)$ in the adapted basis fixed for A is an upper triangular matrix of $Gl_n(\mathbb{F}_p)$ with identity on the diagonal, namely

$$L(y) := \begin{pmatrix} 1 & \ell_{1,2}(y) & \ell_{1,3}(y) & \cdots & \ell_{1,n}(y) \\ 0 & 1 & \ell_{2,3}(y) & \cdots & \ell_{2,n}(y) \\ 0 & 0 & \cdots & \cdots & \ell_{i,n}(y) \\ 0 & 0 & 0 & 1 & \ell_{n-1,n}(y) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Gl_n(\mathbb{F}_p)$$

where, for all i in $\{1, \dots, n-1\}$, $\ell_{i,i+1}$ is a nonzero linear form from V to \mathbb{F}_p (see [Ro08a] section 2.4). In other words,

$$\forall y \in V, f_1(X+y) - f_1(X) = 0 \mod \wp(k[X])$$

$$\forall i \in \{2, \dots, n\}, \forall y \in V, f_i(X+y) - f_i(X) = \sum_{j=1}^{i-1} \ell_{j,i}(y) f_j(X) \mod \wp(k[X]) \quad (9)$$

For all map ℓ , we write $\ell = 0$ if ℓ is identically zero and $\ell \neq 0$ otherwise.

3. The case of a trivial representation can be charactrized as follows (see [Ro08a] Prop. 2.13). Indeed, the following assertions are equivalent:

(a) The representation ρ is trivial, i.e.

$$\forall i \in \{1, \dots, n\}, \quad \forall y \in V, \quad f_i(X+y) - f_i(X) = 0 \mod \wp(k[X])$$

(b) The commutator subgroup of G' and G is trivial, i.e. $G' \subset Z(G)$.

(c) For all i in $\{1, \dots, n\}$, $f_i(X) = X S_i(X) + c_i X \in k[X]$ where each $S_i \in k\{F\}$ is an additive polynomial with degree $s_i \geq 1$ in F . So, write $S_i(F) = \sum_{j=0}^{s_i} a_{i,j} F^j$ with $a_{i,s_i} \neq 0$. Then, one defines an additive polynomial related to f_i , called the "palindromic polynomial" of f_i :

$$Ad_{f_i} := \frac{1}{a_{i,s_i}^{p^{s_i}}} F^{s_i} \left(\sum_{j=0}^{s_i} a_{i,j} F^j + F^{-j} a_{i,j} \right)$$

In this case,

$$V \subset \bigcap_{i=1}^n Z(Ad_{f_i})$$

Since, under condition (*), G' is p -elementary abelian, we deduce from point (b) that the case of a trivial representation corresponds to the case $\{e\} = \text{Fratt}(G') = [G', G]$.

4. To conclude, we recall that for all $t \geq 1$, Σ_t means the k -subvector space of $k[X]$ generated by 1 and the products of at most t additive polynomials of $k[X]$ (cf. [Ro08a] Def. 3.1). As proved in [Ro08a] (Thm. 3.13), for all i in $\{1, \dots, n\}$, f_i lies in Σ_{i+1} .

5.2 First case: big actions satisfying \mathcal{G}_*^p .

Proposition 5.3. We keep the notations of section 5.1.

1. (C, G) is a big action with $G_2 \simeq \mathbb{Z}/p\mathbb{Z}$ if and only if C is birational to a curve C_f parametrized by $W^p - W = f(X) = X S(X) \in k[X]$, where S is a (monic) additive polynomial with degree $s \geq 1$ in F .
2. In what follows, we assume that C is birational to a curve C_f as described in the first point.
 - (a) If $s \geq 2$, $A_{\infty,1}$ is the unique p -Sylow subgroup of A , where ∞ denotes the point of C corresponding to $X = \infty$.

(b) If $s = 1$, there exists $r := p^3 + 1$ points of C : $P_0 := \infty, P_1, \dots, P_r$ such that $(A_{P_i,1})_{0 \leq i \leq r}$ are the p -Sylow subgroups of A . In this case, for all i in $\{1, \dots, r\}$, there exists $\sigma_i \in A$ such that $\sigma_i(P_i) = \infty$.

In both cases, $A_{\infty,1}$ is an extraspecial group (see [Su86] Def. 4.14) with exponent p (resp. p^2) if $p > 2$ (resp. $p = 2$) and order p^{2s+1} . More precisely, $A_{\infty,1}$ is a central extension of its center $Z(A_{\infty,1}) = (A_{\infty,1})'$ by the elementary abelian p -group $Z(Ad_f)$, i.e.

$$0 \longrightarrow Z(A_{\infty,1}) = (A_{\infty,1})' \simeq \mathbb{Z}/p\mathbb{Z} \longrightarrow A_{\infty,1} \xrightarrow{\pi} Z(Ad_f) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s} \longrightarrow 0$$

Furthermore, $(C, A_{\infty,1})$, and so each $(C, A_{P_i,1})$, with $1 \leq i \leq r$, are big actions satisfying \mathcal{G}_*^p .

3. Let V be a subvector space of $Z(Ad_f)$ with dimension v over \mathbb{F}_p . Then, $(C, \pi^{-1}(V))$ is also a big action satisfying \mathcal{G}_*^p if and only if

$$\begin{aligned} \text{if } p \neq 2, \quad & 2s \geq v \geq \max\{s+1, 2s-3\} \\ \text{if } p = 2, \quad & 2s \geq v \geq \max\{s+1, 2s-4\} \end{aligned}$$

We collect the different possibilities in the table below:

case	v	s	V	G
1-	$2s$	$s \geq 1^\dagger$	$Z(Ad_f)^\dagger$	$A_{\infty,1}^\dagger$
2	$2s-1$	$s \geq 2$	index p subgroup of $Z(Ad_f)$	index p subgroup of $A_{\infty,1}$
3	$2s-2$	$s \geq 3$	index p^2 subgroup of $Z(Ad_f)$	index p^2 subgroup of $A_{\infty,1}$
4	$2s-3$	$s \geq 4$	index p^3 subgroup of $Z(Ad_f)$	index p^3 subgroup of $A_{\infty,1}$
5 ($p=2$)	$2s-4$	$s \geq 5$	index p^4 subgroup of $Z(Ad_f)$	index p^4 subgroup of $A_{\infty,1}$

case	$ G /g$	$ G /g^2$
1	$\frac{2p}{p-1} p^s$	$\frac{4}{(p^2-1)^2} (p+1)^2 p$
2	$\frac{2p}{p-1} p^{s-1}$	$\frac{4}{(p^2-1)^2} (p+1)^2$
3	$\frac{2p}{p-1} p^{s-2}$	$\frac{4}{(p^2-1)^2} \frac{(p+1)^2}{p}$
4	$\frac{2p}{p-1} p^{s-3}$	$\frac{4}{(p^2-1)^2} \frac{(p+1)^2}{p^2}$
5 ($p=2$)	$\frac{2p}{p-1} p^{s-4}$	$\frac{4}{(p^2-1)^2} \frac{(p+1)^2}{p^3}$

\dagger Note: In the case $s = 1$, this result is true up to conjugation by σ_i as defined in Proposition 5.3.

Proof:

1. See [LM05] (Thm. 1.1 I)
2. See Remark 3.1, [LM05] (Thm. 3.1) and [MR08] (Prop. 2.5).
3. This essentially derives from Proposition 3.2 which implies $(p+1)^2 \geq p^{2s-v-1}$. If $2s-v-1 \geq 3$, it implies $p^2 + 2p + 1 \geq p^3$, which is impossible for $p > 2$. Accordingly, if $p > 2$, we obtain $2s-v-1 \leq 2$, which means $v \geq 2s-3$. If $p = 2$, $(p+1)^2 \geq p^{2s-v-1}$ is satisfied if and only if $2s-v-1 \leq 3$, i.e. $v \geq 2s-4$. The claim follows. \square

Remark 5.4. Note that, for $p > 2$, the solutions can be parametrized by s algebraically independent variables over \mathbb{F}_p , namely the s coefficients of S assumed monic after an homothety on the variable X . Note that $s \sim \log g$.

5.3 Second case: big actions satisfying $\mathcal{G}_*^{p^2}$.

5.3.1 Case: $[G', G] = \text{Fratt}(G') = \{e\}$.

Proposition 5.5. Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^2}$. Assume that $[G', G] = \{e\}$ and keep the notations of section 5.1.

1. The pair $(C, A_{\infty,1})$ is a big action satisfying $\mathcal{G}_*^{p^2}$. Moreover, $A_{\infty,1}$ is a special group (see [Su86] Def. 4.14) with exponent p (resp. p^2) (for $p > 2$ (resp. $p = 2$) and order p^{2+2s_1} . More precisely, $A_{\infty,1}$ is a central extension of its center $Z(A_{\infty,1}) = (A_{\infty,1})'$ by the elementary abelian p -group $Z(\text{Ad}_{f_1})$, i.e.

$$0 \longrightarrow Z(A_{\infty,1}) = (A_{\infty,1})' \simeq (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow A_{\infty,1} \xrightarrow{\pi} Z(\text{Ad}_{f_1}) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0$$

2. Furthermore, $s_2 = s_1$ or $s_2 = s_1 + 1$.

- (a) If $s_2 = s_1$, $G = \pi^{-1}(V)$, where V is a subvector space of $Z(\text{Ad}_{f_1})$ with dimension v over \mathbb{F}_p such that $2s_1 - 2 \leq v \leq 2s_1$. Then, $A_{\infty,1}$ is a p -Sylow subgroup of A . It is normal except if C is birational to the Hermitian curve: $W^q - W = X^{1+q}$ with $q = p^2$.
- (b) If $s_2 = s_1 + 1$, $V = Z(\text{Ad}_{f_1})$ and $G = A_{\infty,1}$ is the unique p -Sylow subgroup of A .

The different possibilities are listed in the table below:

case	s_1	s_2	v	V	G
(a)-1	$s \geq 2$	s	$2s$	$Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2})$	$A_{\infty,1}$
(a)-2	$s \geq 2$	s	$2s - 1$	index p subgroup of $Z(\text{Ad}_{f_1})$	index p subgroup of $A_{\infty,1}$
(a)-3	$s \geq 3$	s	$2s - 2$	index p^2 subgroup of $Z(\text{Ad}_{f_1})$	index p^2 subgroup of $A_{\infty,1}$
(b)	$s \geq 3$	$s + 1$	$2s$	$Z(\text{Ad}_{f_1})$	$A_{\infty,1}$

case	$ G /g$	$ G /g^2$
(a)-1	$\frac{2p}{p-1} \frac{p^{1+s}}{1+p}$	$\frac{4}{(p^2-1)^2} p^2$
(a)-2	$\frac{2p}{p-1} \frac{p^s}{1+p}$	$\frac{4}{(p^2-1)^2} p$
(a)-3	$\frac{2p}{p-1} \frac{p^{s-1}}{1+p}$	$\frac{4}{(p^2-1)^2}$
(b)	$\frac{2p}{p-1} \frac{p^{1+s}}{1+p}$	$\frac{4}{(p^2-1)^2} \frac{p^2(p+1)^2}{(1+p^2)^2}$

Proof:

1. Use Proposition 3.2 to prove that the pair $(C, A_{\infty,1})$ is a big action satisfying $\mathcal{G}_*^{p^2}$ with the following exact sequence:

$$0 \longrightarrow A_{\infty,2} \longrightarrow A_{\infty,1} \xrightarrow{\pi} Z(\text{Ad}_{f_1}) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0$$

The proof to show that $A_{\infty,1}$ is a special group, i.e. satisfies $Z(A_{\infty,1}) = (A_{\infty,1})' = \text{Fratt}(A_{\infty,1}) \simeq (\mathbb{Z}/p\mathbb{Z})^2$, is the same that the one exposed in [Ro08a] (Prop. 4.3.3). Nevertheless, one has to choose \mathcal{H} an index p -subgroup of G_2 such that C/\mathcal{H} is the curve parametrized by $W_1^p - W_1 = f_1(X)$.

2. Assume that $s_2 - s_1 \geq 2$. Then, $|G| = p^{2+v} \leq p^{2+2s_1}$ and $g = \frac{p-1}{2} p^{s_1} (1 + p^{1+s_2-s_1}) \geq \frac{p-1}{2} p^{s_1} (1 + p^3)$. So, $\frac{|G|}{g^2} \leq \frac{4}{(p^2-1)^2} \frac{(1+p)^2 p^2}{(1+p^3)^2} < \frac{4}{(p^2-1)^2}$, which contradicts condition (*). So, $0 \leq s_2 - s_1 \leq 1$. In each case, the description of $A_{\infty,1}$ and G derive from Proposition 3.2 combined with Remark 3.1. \square

To go further in the description of the functions $f'_i s$ in each case, we introduce two additive polynomials \mathcal{V} and T defined as follows:

$$\forall i \in \{1, 2\}, \quad \mathcal{V} := \prod_{y \in V} (X - y) \quad \text{divides} \quad T := \gcd\{\text{Ad}_{f_1}, \text{Ad}_{f_2}\} \quad \text{divides} \quad \text{Ad}_{f_i}$$

In what follows, we work in the Ore ring $k\{F\}$ and write the additive polynomials as polynomials in F .

case	$\deg_F \mathcal{V}$	$\deg_F T$	$\deg_F(\text{Ad}_{f_1})$	$\deg_F(\text{Ad}_{f_2})$	T
(a)-1	$2s$	$2s$	$2s$	$2s$	$\mathcal{V} = T = \text{Ad}_{f_1} = \text{Ad}_{f_2}$
(a)-2-i	$2s - 1$	$2s$	$2s$	$2s$	\mathcal{V} divides $T = \text{Ad}_{f_1} = \text{Ad}_{f_2}$
(a)-2-ii	$2s - 1$	$2s - 1$	$2s$	$2s$	$\mathcal{V} = T$ divides Ad_{f_1}
(a)-3-i	$2s - 2$	$2s$	$2s$	$2s$	\mathcal{V} divides $T = \text{Ad}_{f_1} = \text{Ad}_{f_2}$
(a)-3-ii	$2s - 2$	$2s - 1$	$2s$	$2s$	\mathcal{V} divides T divides Ad_{f_1}
(a)-3-iii	$2s - 2$	$2s - 2$	$2s$	$2s$	$\mathcal{V} = T$ divides Ad_{f_1}
(b)	$2s$	$2s$	$2s$	$2s + 2$	$\mathcal{V} = T = \text{Ad}_{f_1}$ divides Ad_{f_2}

The three cases where $Ad_{f_1} = Ad_{f_2}$ can be parametrized in the same way as in [Ro08a] (Prop. 4.2).

case	S_1 or Ad_{f_1}	S_2 or Ad_{f_2}
(a)-1	$S_1 = \sum_{j=0}^{s/d} \alpha_{jd} F^{jd}, \quad \alpha_s = 1$	$S_2 = \gamma S_1, \quad \gamma \in \mathbb{F}_{p^d} - \mathbb{F}_p, \quad d \geq 2$
(a)-2-i	$S_1 = \sum_{j=0}^{s/d} \alpha_{jd} F^{jd}, \quad \alpha_s = 1$	$S_2 = \gamma S_1, \quad \gamma \in \mathbb{F}_{p^d} - \mathbb{F}_p, \quad d \geq 2$
(a)-2-ii	$Ad_{f_1} = (\alpha_1 F + \beta_1 I) T, \quad \alpha_1 \neq 0$	$Ad_{f_2} = (\alpha_2 F + \beta_2 I) T, \quad \alpha_2 \neq 0$
(a)-3-i	$S_1 = \sum_{j=0}^{s/d} \alpha_{jd} F^{jd}, \quad \alpha_s = 1$	$S_2 = \gamma S_1, \quad \gamma \in \mathbb{F}_{p^d} - \mathbb{F}_p, \quad d \geq 2$
(a)-3-ii	$Ad_{f_1} = (\alpha_1 F + \beta_1 I) T, \quad \alpha_1 \neq 0$	$Ad_{f_2} = (\alpha_2 F + \beta_2 I) T, \quad \alpha_2 \neq 0$
(a)-3-iii	$Ad_{f_1} = (\alpha_1 F^2 + \beta_1 F + \delta_1 I) T, \quad \alpha_1 \neq 0$	$Ad_{f_2} = (\alpha_2 F^2 + \beta_2 F + \delta_2 I) T, \quad \alpha_2 \neq 0$
(b)	$Ad_{f_1} = \prod_{v \in V} (X - v)$	$Ad_{f_2} = (\alpha_2 F^2 + \beta_2 F + \delta_2 I) Ad_{f_1}, \quad \alpha_2 \neq 0$

We display the parametrization of the functions f_i 's in the case (a)-2-ii for the smallest values of s , namely $s = 2$ and $s = 3$.

Cas (a)-2-ii with $s = 2$ for $p > 2$.

f_1	$f_1(X) = X^{1+p^2} + a_{1+p} X^{1+p} + \frac{1}{2} a_2 X^2$
a_{1+p}	$a_{1+p} \in k$
a_2	$a_2 \in k$
f_2	$f_2(X) = b_{1+p^2}^2 X^{1+p^2} + b_{1+p} X^{1+p} + b_2 X^2 + b_1 X$
b_{1+p^2}	$b_{1+p^2} \in Z(w^{p^2} X^{p^3} + w^p(-a_2^p + a_{1+p}^p w^{p^2} - w^{p^2+p^3}) X^{p^2} + (a_{1+p} - w^{p^2}) X^p - w^{-1} X)$ with $b_{1+p^2} \notin \mathbb{F}_{p^2}$.
w	$w \in Z(X^{1+p+p^2+p^3} - a_{1+p}^p X^{1+p+p^2} + a_2^p X^{1+p} - a_{1+p} X + 1)$
b_{1+p}	$b_{1+p} = w^{p^2} (b_{1+p^2}^{p^2} - b_{1+p^2})^p + b_{1+p^2}^p a_{1+p}$
b_2	$2 b_2 = w^p (b_{1+p^2}^{p^2} - b_{1+p^2}) (a_{1+p} - w^{p^2}) + b_{1+p^2}^2$
b_1	$b_1 \in k$

Case (a)-2-ii with $s = 3$ for $p > 2$.

f_1	$f_1(X) = X^{1+p^3} + a_{1+p^2} X^{1+p^2} + a_{1+p} X^{1+p} + \frac{1}{2} a_2 X^2$
a_{1+p^2}	$a_{1+p^2} \in k$
a_{1+p}	$a_{1+p} \in k$
a_2	$a_2 \in k$
f_2	$f_2(X) = b_{1+p^3}^3 X^{1+p^3} + b_{1+p^2} X^{1+p^2} + b_{1+p} X^{1+p} + b_2 X^2 + b_1 X$
w	$w \in Z(X^{1+p+p^2+p^3+p^4+p^5} - a_{1+p^2}^{p^2} X^{1+p+p^2+p^3+p^4} + a_{1+p}^{p^2} X^{1+p+p^2+p^3} - a_2^{p^2} X^{1+p+p^2} + a_{1+p}^p X^{1+p} - a_{1+p^2} X + 1)$
b_{1+p^3}	$b_{1+p^3} \in Z(P_1) \cap Z(P_2) - \mathbb{F}_{p^3}$ with $P_1(X) = w^{p^3+1} X^{p^5} + (1 - w a_{1+p^2}) X^{p^3} + (w a_{1+p^2} - w^{p^3+1}) X^{p^2} - X$ with $P_2(X) = w^{p^2} (a_{1+p^2} - w^{p^3}) X^{p^4} + w^p (-a_2^p + a_{1+p}^p w^{p^2} - a_{1+p^2}^p w^{p^2+p^3} + w^{p^2+p^3+p^4}) X^{p^3} + (a_{1+p} + w^{p^2+p^3} - a_{1+p^2} w^{p^2}) X^p + (-a_{1+p} + a_2^p w^p - a_{1+p}^p w^{p+p^2} + a_{1+p^2}^p w^{p+p^2+p^3} - w^{p+p^2+p^3+p^4}) X$
b_{1+p^2}	$b_{1+p^2} = w^{p^3} (b_{1+p^3}^{p^3} - b_{1+p^3})^{p^2} + b_{1+p^3}^{p^2} a_{1+p^2}$
b_{1+p}	$b_{1+p} = w^{p^2} (b_{1+p^3}^{p^3} - b_{1+p^3})^p (a_{1+p^2} - w^{p^3}) + b_{1+p^3}^p a_{1+p}$
b_2	$2 b_2 = w^p (b_{1+p^3}^{p^3} - b_{1+p^3}) (a_{1+p} - a_{1+p^2} w^{p^2} + w^{p^2+p^3}) + b_{1+p^3} a_2$
b_1	$b_1 \in k$

The calculation of the case $s = 3$ already raises a problem as the parameter b_{1+p^3} has to lie in the set of zeroes of two polynomials.

For the remaining last two cases (a)-3-iii and (b), we merely display examples of realization so as to prove the effectiveness of these cases.

An example of realization for the case (a)-3-iii.

T	$T = F^{2s-2} + I$
V	$V = Z(F^{2s-2} + I)$
Ad_{f_1}	$Ad_{f_1} = (F^2 + I) T$
f_1	$f_1(X) = X^{1+p^s} + X^{1+p^{s-2}}$
Ad_{f_2}	$Ad_{f_2} = (F^2 + F + I) T$
f_2	$f_2(X) = X^{1+p^s} + X^{1+p^{s-1}} + X^{1+p^{s-2}}$

An example of realization for the case (b).

f_1	$f_1(X) = X^{1+p^s}$
f_2	$f_2(X) = \alpha_2 X^{1+p^{s+1}} + \beta_2 X^{1+p^s} + \delta_2 X^{1+p^{s-1}}$
α_2	$\alpha_2 \in \mathbb{F}_{p^{2s}}$
β_2	$\beta_2 \in \mathbb{F}_{p^s}$
δ_2	$\delta_2 \in \mathbb{F}_{p^{2s}}$

5.3.2 Case: $[G', G] \supsetneq \text{Fratt}(G') = \{e\}$.

Proposition 5.6. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^2}$ such that $[G', G] \neq \{e\}$. We keep the notations introduced in section 5.1.*

1. (a) Then, $G = A_{\infty,1}$ is the unique p -Sylow subgroup of A .
(b) For all i in $\{1, 2\}$, $f_i \in \Sigma_{i+1} - \Sigma_i$ and $m_i = 1 + i p^s$, with $p \geq 3$ and $s \in \{1, 2\}$.
(c) Moreover, $v = s + 1$. More precisely, $y \in V$ if and only if $\ell_{1,2}(y)^p - \ell_{1,2}(y) = 0$.
2. There exists a coordinate X for the projective line C/G_2 such that the functions f_i 's are parametrized as follows:

(a) If $s = 1$,

	$p > 3$	$p = 3$
f_1	$f_1(X) = X^{1+p} + a_2 X^2$	$f_1(X) = X^4 + a_2 X$
V	$V = Z(Ad_{f_1}) = Z(X^{p^2} + 2 a_2^p X^p + X)$	$V = Z(Ad_{f_1}) = Z(X^9 + 2 a_2^3 X^3 + X)$
f_2	$f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_3 X^3 + b_1 X$	$f_2(X) = b_7 X^7 + b_5 X^5 + b_1 X$
b_{1+2p}	$b_{1+2p} \in k^\times$	$b_7^{16} = 1$
a_2	$2 a_2^p = -b_{1+2p}^{-p} (b_{1+2p}^{p^2} + b_{1+2p}) \Leftrightarrow b_{1+2p} \in V$	$2 a_2^3 = -b_7^6 - b_7^{-2}$
b_{2+p}	$b_{2+p} = -b_{1+2p}^p$	$b_5 = -b_7^3$
b_3	$3 b_3^p = b_{1+2p}^{-p} (b_{1+2p}^{2p^2} - b_{1+2p}^2)$	
b_1	$b_1 \in k$	$b_1 \in k$
$\ell_{1,2}$	$\ell_{1,2}(y) = 2 (b_{1+2p} y^p - b_{1+2p}^p y)$	$\ell_{1,2}(y) = 2 (b_7 y^3 - b_7^3 y)$

Therefore, for $p > 3$, the solutions are parametrized by 2 algebraically independent variables over \mathbb{F}_p , namely $b_{1+2p} \in k^\times$ and $b_1 \in k$. For $p = 3$, as the monomial X^3 can be reduced mod $\wp(k[X])$, the parameter b_{1+2p} satisfies an additional algebraic relation: $b_7^{16} = 1$. Then, b_7 takes a finite number of values.

In both cases ($p = 3$ or $p > 3$),

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^2}{1+2p} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2(p+1)^2}{(1+2p)^2}$$

(b) If $s = 2$ and $p > 3$,

f_1	$f_1(X) = X^{1+p^2} + a_{1+p} X^{1+p} + a_2 X^2$
Ad_{f_1}	$X^{p^4} + a_{1+p}^2 X^{p^3} + 2 a_2^2 X^{p^2} + a_{1+p}^p X^p + X$
f_2	$f_2(X) = b_{1+2p^2} X^{1+2p^2} + b_{1+p+p^2} X^{1+p+p^2} + b_{2+p^2} X^{2+p^2} + b_{1+p^2} X^{1+p^2} + b_{1+2p} X^{1+2p}$ $+ b_{2+p} X^{2+p} + b_{1+p} X^{1+p} + b_3 X^3 + b_2 X^2 + b_1 X$
b_{1+2p}	$b_{1+2p} \in k^\times$
b_{2+p^2}	$b_{2+p^2} \in k^\times$
b_{1+p+p^2}	$b_{1+p+p^2}^p = -2 b_{1+2p}^p (b_{2+p^2}^p b_{1+2p}^{-p^2} + b_{2+p^2}^{p-1})$
$\ell_{1,2}$	$\forall y \in V, \quad \ell_{1,2}(y) = 2 b_{1+2p} y^{p^2} + b_{1+p+p^2} y^p + 2 b_{2+p^2} y$
V	V is an index p -subgroup of $Z(Ad_{f_1})$ $V = Z(2 b_{1+2p}^p X^{p^3} + (b_{1+p+p^2}^p - 2 b_{1+2p}) X^{p^2} + (2 b_{2+p^2}^p - b_{1+p+p^2}) X^p - 2 b_{2+p^2} X)$
a_{1+p}	$a_{1+p}^p = -b_{1+2p}^{p-p^2} - b_{1+2p}^p b_{2+p^2}^{-1} - b_{2+p^2}^{p^2} b_{1+2p}^{-p^3} - b_{2+p^2}^{p^2-p}$
a_2	$2 a_2^p = b_{2+p^2}^{p^2} b_{1+2p}^{-p^2} + b_{1+2p} b_{2+p^2}^{-1} + b_{2+p^2}^p b_{1+2p}^{p-2p^2} + 2 b_{2+p^2}^{p-1} b_{1+2p}^{p-p^2} + b_{1+2p}^p b_{2+p^2}^{p-2}$
b_{1+2p}	$b_{1+2p}^{p^2} = -b_{1+2p}^{2p-p^2} - b_{1+2p}^{2p} b_{2+p^2}^{-1} + b_{2+p^2}^{2p^2} b_{1+2p}^{p^2-2p^3} + 2 b_{2+p^2}^{2p^2-p} b_{1+2p}^{p^2-p^3} + b_{1+2p}^{p^2} b_{2+p^2}^{2p^2-2p}$
b_{2+p}	$b_{2+p}^{p^2} = b_{2+p^2}^p b_{1+2p}^{2p-2p^2} + 2 b_{2+p^2}^{p-1} b_{1+2p}^{2p-p^2} + b_{1+2p}^{2p} b_{2+p^2}^{p-2} - b_{2+p^2}^{2p^2} b_{1+2p}^{-p^3} - b_{2+p^2}^{2p^2-p}$
b_3	$3 b_3^{p^2} = b_{2+p^2}^{2p^2} b_{1+2p}^{-p^2} - b_{2+p^2}^{2p} b_{1+2p}^{2p-3p^2} - 3 b_{2+p^2}^{2p-2} b_{1+2p}^{2p-p^2} - 3 b_{2+p^2}^{2p-1} b_{1+2p}^{2p-2p^2} - b_{1+2p}^{2p} b_{2+p^2}^{2p-3} + b_{1+2p}^2 b_{2+p^2}^{-1}$
b_{1+p^2}	$b_{1+p^2} \in Z(b_{2+p^2}^{p^2} b_{1+2p}^{-p^3} X^{p^3} - (b_{2+p^2}^{p^2} b_{1+2p}^{-p^3} + b_{1+2p}^{p-p^2} + b_{2+p^2}^{p^2-p}) X^{p^2} +$ $(b_{1+2p}^{p-p^2} + b_{1+2p}^p b_{2+p^2}^{-1} + b_{2+p^2}^{p^2-p}) X^p - b_{1+2p}^p b_{2+p^2}^{-1} X)$
b_{1+p}	$b_{1+p}^{p^2} = -(b_{1+2p}^{p-p^2} + b_{2+p^2}^{p^2} b_{1+2p}^{-p^3} + b_{2+p^2}^{p^2-p}) b_{1+p^2}^{p^2} - b_{1+2p}^p b_{2+p^2}^{-1} b_{1+p^2}$
b_2	$2 b_2^{p^2} = (b_{2+p^2}^p b_{1+2p}^{p-2p^2} + b_{2+p^2}^{p-1} b_{1+2p}^{p-p^2-p} + b_{2+p^2}^{p^2} b_{1+2p}^{-p^2}) b_{1+p^2}^p +$ $(b_{1+2p}^p b_{2+p^2}^{p-2} + b_{2+p^2}^{p-1} b_{1+2p}^{p-p^2} + b_{1+2p} b_{2+p^2}^{-1}) b_{1+p^2}$
b_1	$b_1 \in k$

Therefore, for $p > 3$, the solutions can be parametrized by 3 algebraically independent variables over \mathbb{F}_p , namely $b_{1+2p^2} \in k^\times$, $b_{2+p^2} \in k^\times$ and $b_1 \in k$. One also finds a fourth parameter b_{1+p^2} which runs over an \mathbb{F}_p -subvector space of k , namely the set of zeroes of an additive separable polynomial whose coefficients are rational functions in b_{1+2p^2} and b_{2+p^2} . So, for given b_{1+2p} and b_{2+p^2} , the parameter b_{1+p^2} takes a finite number of values.

For $p = 3$,

$f_1(X) =$	$X^{10} + a_4 X^4 + a_2 X^2$
$f_2(X) =$	$b_{19} X^{19} + b_{13} X^{13} + b_{11} X^{11} + b_{10} X^{10} + b_7 X^7 + b_5 X^5 +$ $b_4 X^4 + b_2 X^2 + b_1 X$

with a_4 , a_2 , b_{13} , b_7 , b_5 , b_3 and b_2 satisfying the same relations as above. But, this time, the parameters b_{19} and b_{11} are linked through an algebraic relation, namely:

$$b_{11}^{18} b_{19}^{-9} - b_{11}^6 b_{19}^{-21} - b_{19}^6 b_{11}^3 + b_{19}^2 b_{11}^{-1} = 0$$

In both cases ($p = 3$ or $p > 3$),

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^2}{1+2p} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p(p+1)^2}{(1+2p)^2}$$

Remark 5.7. One can now answer the second problem raised in [Ro08a] (section 6). Indeed, one notices that the family obtained for $s = 2$ is larger than the one obtained after the additive base change: $X = Z^p + cZ$, $c \in k - \{0\}$ (see [MR08] Prop. 3.1) applied to the case $s = 1$. Indeed, such a base change does not produce any monomial Z^{1+p^2} in $f_2(Z)$.

A few special cases.

1. When $s = 1$ and $p > 3$, the special case $a_2 = 0$ corresponds to the parametrization of the extension K_S^m/K given by Auer (cf. [Au99] Prop. 8.9 or [MR08] section 6), namely

$$f_1(X) = a X^{1+p} \text{ with } a^p + a = 0, a \neq 0.$$

$$f_2(X) = a^2 X^{2p} (X - X^{p^2}).$$

2. When $s = 2$, the special case $b_{1+p^2} \in \mathbb{F}_p$ leads to $b_{1+p} = b_{1+p^2} a_{1+p}$ and $b_2 = b_{1+p^2} a_2$. So, one can replace f_2 by $f_2(X) - b_{1+p^2} f_1(X)$, which eliminates the monomials X^{1+p^2} , X^{1+p} and X^2 .

Proof of Proposition 5.6:

1. As $\ell_{1,2} \neq 0$, the group G satisfies the third condition of [Ro08a] (Prop. 5.2). Then, the equality $G = A_{\infty,1}$ derives from [Ro08a] (Cor. 5.7). The unicity of the p -Sylow subgroup is explained in Remark 3.1. The second and third assertions come from [Ro08a] (Thm. 5.6). Moreover, the description of V displayed in (c) is due to [Ro08a] (Prop 2.9.2). It remains to show that $s = 1$ or $s = 2$. Using formula (8), we compute $g = \frac{(p-1)}{2} (p^s + p(m_2 - 1)) = \frac{(p-1)}{2} p^s (1 + 2p)$. As $|G| = p^{3+s}$, condition (*) requires: $\frac{4}{(p^2-1)^2} \leq \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{(p+1)^2}{p^{s-3}(1+2p)^2}$. It follows that $3 - s > 0$, i.e. $1 \leq s \leq 2$.
2. We merely explain the case $s = 1$. One can find a coordinate X of the projective line C/G_2 such that $f_1(X) = X S_1(X) = X(X^p + a_2 X)$ (cf. [Ro08a] Cor. 2.12). Then, $Ad_{f_1} = F^2 + 2a_2^p F + I$ (cf. [Ro08a] Prop. 2.13). As $V \subset Z(Ad_{f_1})$ and $\dim_{\mathbb{F}_p} Z(Ad_{f_1}) = 2 = s + 1 = v$, we deduce that $V = Z(Ad_{f_1})$. As $f_2 \in \Sigma_3 - \Sigma_2$ with $\deg f_2 = 1 + 2p^s$ and as the functions f_i 's are supposed to be reduced mod $\wp(k[X])$, equation (9) reads:

$$\forall y \in V, \quad f_2(X+y) - f_2(X) = \ell_{1,2}(y) f_1(X) \quad \text{mod } \wp(k[X])$$

$$\begin{aligned} &\text{with} \quad f_1(X) = X^{1+p} + a_2 X^2 \\ &\text{and} \quad f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_{1+p} X^{1+p} + b_3 X^3 + b_2 X^2 + b_1 X \quad \text{for } p > 3 \\ &(\text{resp.} \quad f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_{1+p} X^{1+p} + b_2 X^2 + b_1 X \quad \text{for } p = 3) \end{aligned}$$

Then, calculation gives the relations gathered in the table. In particular, we find: $f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_3 X^3 + b_1 X + b_{1+p} f_1(X)$ with $b_{1+p} \in \mathbb{F}_p$. Since we are working in the \mathbb{F}_p -space generated by $f_1(X)$ and $f_2(X)$, we can replace $f_2(X)$ with $f_2(X) - b_{1+p} f_1(X)$, hence the expected formula. We solve the case $s = 2$ in the same way. \square

5.4 Third case: big actions satisfying $\mathcal{G}_*^{p^3}$.

5.4.1 Preliminaries.

The idea is to use, as often as possible, the results obtained in the preceding section.

Remark 5.8. Let (C, G) be a big action with $G' (= G_2) \simeq (\mathbb{Z}/p\mathbb{Z})^3$. We keep the notations introduced in section 5.1.

1. Let $C_{1,2}$ be the curve parametrized by the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{1, 2\}$, and let $K_{1,2} := k(C_{1,2})$ be the function field of this curve. Then, $K_{1,2}/k(X)$ is a Galois extension with group $\Gamma_{1,2} \simeq (\mathbb{Z}/p\mathbb{Z})^2$. Moreover, the group of translations by $V: \{X \rightarrow X + y, y \in V\}$ extends to an automorphism p -group of $C_{1,2}$ say $G_{1,2}$, with the following exact sequence:

$$0 \longrightarrow \Gamma_{1,2} \longrightarrow G_{1,2} \longrightarrow V \longrightarrow 0$$

Let $A_{1,2}$ be the \mathbb{F}_p -subvector space of A generated by the classes of $f_1(X)$ and $f_2(X)$. Let $H_{1,2} \subsetneq G_2$ be the orthogonal of $A_{1,2}$ with respect to the Artin-Schreier pairing. Then, $C_{1,2} = C/H_{1,2}$ and $G_{1,2} = G/H_{1,2}$. Furthermore, as $A_{1,2}$ is stable under the action of ρ , its dual $H_{1,2}$ is stable by the dual representation ϕ , i.e. by conjugation by the elements of G (see section 5.1). It follows that $H_{1,2} \subsetneq G_2$ is a normal subgroup in G . So, by [MR08] (Lemma 2.4), the pair $(C_{1,2}, G_{1,2})$ is a big action with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

2. Likewise, if $\ell_{2,3} = 0$, the \mathbb{F}_p -subvector space of A generated by the classes of $f_1(X)$ and $f_3(X)$ is also stable by ρ (see matrix $L(y)$ in section 5.1). So, the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{1, 3\}$, also parametrize a big action, say $(C_{1,3}, G_{1,3})$, with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.
3. Similarly, if $\ell_{1,2} = \ell_{1,3} = 0$, the \mathbb{F}_p -subvector space of A generated by the classes of $f_2(X)$ and $f_3(X)$ is stable by ρ (see matrix $L(y)$ in section 5.1). So, the two equations: $W_i^p - W_i = f_i(X)$, with $i \in \{2, 3\}$, also parametrize a big action, say $(C_{2,3}, G_{2,3})$, with second ramification group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

Lemma 5.9. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^2}$. Let $(C_{1,2}, G_{1,2})$ be defined as in Remark 5.8. We keep the notations introduced in section 5.1.*

1. *Then, $(C_{1,2}, G_{1,2})$ is a big action satisfying $\mathcal{G}_*^{p^2}$.*
2. *If $\ell_{1,2} = 0$, then $m_1 = m_2 = 1 + p^s$, with $s \geq 2$.*
3. *If $\ell_{1,2} \neq 0$, then $m_1 = 1 + p^s$, $m_2 = 1 + 2p^s$, with $s \in \{1, 2\}$ and $p \geq 3$. In this case, $v = s + 1$.*

Proof:

1. Use Remark 5.8 and [LM05] (Prop. 8.5 (ii)) to see that condition $(*)$ is still satisfied.
2. We deduce from Proposition 5.5 that $m_1 = 1 + p^{s_1}$ and $m_2 = 1 + p^{s_2}$ with $s_2 = s_1$ or $s_2 = s_1 + 1$. Assume that $s_2 = s_1 + 1$. Then, $m_3 \geq m_2 = 1 + p^{s_1+1}$. We compute the genus by means of (8): $g = \frac{p-1}{2} (p^{s_1} + p^{1+s_2} + p^2 (m_3 - 1)) \geq \frac{p-1}{2} p^{s_1} (1 + p^2 + p^3)$. Besides, by [MR08] (Thm. 2.6), $V \subset Z(\text{Ad}_{f_1})$, so $|G| = p^{3+v} \leq p^{3+2s_1}$. Thus, $\frac{|G|}{g^2} \leq \frac{4}{(p^2-1)^2} \frac{p^3 (p+1)^2}{(1+p^2+p^3)^2} < \frac{4}{(p^2-1)^2}$, which contradicts condition $(*)$. It follows that $s_2 = s_1 \geq 2$.
3. Apply Proposition 5.6 to (C_{12}, G_{12}) . \square

Remark 5.10. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$. Assume that $\ell_{1,2} = \ell_{1,3} = 0$. Then, the results of Lemma 5.9 also hold for the big action $(C_{2,3}, G_{2,3})$ as defined in Remark 5.8.*

Lemma 5.11. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$. We keep the notations introduced in section 5.1. Assume that $\ell_{2,3} = 0$. Let $(C_{1,3}, G_{1,3})$ be defined as in Remark 5.8.*

1. *Then, $(C_{1,3}, G_{1,3})$ is a big action satisfying $\mathcal{G}_*^{p^2}$.*
2. *If $\ell_{1,3} = 0$, then $\ell_{1,2} = 0$ and $m_1 = m_2 = m_3 = 1 + p^s$ with $s \geq 2$. In this case, $v = 2s$.*
3. *If $\ell_{1,3} \neq 0$, then $m_1 = 1 + p^s$, $m_3 = 1 + 2p^s$, with $s \in \{1, 2\}$ and $p \geq 3$. In this case, $v = s + 1$.*

Proof:

1. Use Remark 5.8 and [LM05] (Prop. 8.5 (ii)).
2. As $\ell_{1,3} = 0$, we deduce from Proposition 5.5 that $m_1 = 1 + p^s$ and $m_3 = 1 + p^{s_3}$ with $s_3 = s$ or $s_3 = s + 1$.
 - (a) *We show that $\ell_{1,2} = 0$.*
Assume that $\ell_{1,2} \neq 0$. Then, Lemma 5.9 applied to $(C_{1,2}, G_{1,2})$ implies $m_2 = 1 + 2p^s$ with $s \in \{1, 2\}$ and $p \geq 3$. Moreover, $v = s + 1$. As $m_2 \leq m_3$, there are two possibilities:
 - i. $s = 1$ and $s_3 = s + 1 = 2$, i.e. $m_1 = 1 + p$, $m_2 = 1 + 2p$, $m_3 = 1 + p^2$ and $v = 2$.
Then, $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3 (p+1)^2}{(1+2p+p^3)^2} < \frac{4}{(p^2-1)^2}$, which contradicts condition $(*)$.
 - ii. $s = 2$ and $s_3 = s + 1 = 3$, i.e. $m_1 = 1 + p^2$, $m_2 = 1 + 2p^2$, $m_3 = 1 + p^3$ and $v = 3$.
Then, $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2 (p+1)^2}{(1+2p+p^3)^2} < \frac{4}{(p^2-1)^2}$, which also contradicts condition $(*)$.
As a consequence, $\ell_{1,2} = 0$.

- (b) *We deduce that $m_1 = m_2 = 1 + p^s$ with $s \geq 2$.*

Lemma 5.9 applied to $(C_{1,2}, G_{1,2})$ implies $m_1 = m_2 = 1 + p^s$ with $s \geq 2$. In particular, $g = \frac{p-1}{2} p^s (1 + p + p^{2+s_3-s})$ and $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^{3+v-2s} (p+1)^2}{(1+p+p^{2+s_3-s})^2}$.

- (c) *We show that $v = 2s_3$ and conclude that $s_3 = s$.*

Assume that $v \leq 2s_3 - 3$. Then, $\frac{|G|}{g^2} < \frac{4}{(p^2-1)^2} \frac{p^{2s_3-2s} (p+1)^2}{p^{4+2s_3-2s}} < \frac{4}{(p^2-1)^2}$ which contradicts condition $(*)$. Therefore, $2s_3 - 2 \leq v \leq 2s \leq 2s_3$. Assume that $v \leq 2s_3 - 3$. Then, $\frac{|G|}{g^2} < \frac{4}{(p^2-1)^2} \frac{p^{2s_3-2s} (p+1)^2}{p^{4+2s_3-2s}} < \frac{4}{(p^2-1)^2}$ which contradicts condition $(*)$. Assume that $v = 2s_3 - 1$. So, v is odd and $2s_3 - 2 < v \leq 2s \leq 2s_3$ implies $s_3 = s$ and $v = 2s - 1$. Then, $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2 (p+1)^2}{(1+p+p^2)^2} < \frac{4}{(p^2-1)^2}$, which is excluded. Now, assume that $v = 2s_3 - 2$. Then, $2s_3 - 2 = v \leq 2s \leq 2s_3$ implies $s_3 = s$ or $s_3 = s + 1$. In the first case, $v = 2s - 2$ and $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p (p+1)^2}{(1+p+p^2)^2} < \frac{4}{(p^2-1)^2}$. In the second case, $v = 2s$ and $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3 (p+1)^2}{(1+p+p^3)^2} < \frac{4}{(p^2-1)^2}$. In both cases, we obtain a contradiction. We gather that $v = 2s_3$. Applying [Ro08a] (Prop. 4.2), we conclude that $s = s_3$.

3. Apply Proposition 5.6 to (C_{13}, G_{13}) . \square

5.4.2 Case: $[G', G] = \text{Fratt}(G') = \{e\}$.

Proposition 5.12. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$ such that $[G', G] = \{e\}$. We keep the notations introduced in section 5.1.*

1. *Then, $G = A_{\infty,1}$ is a special group of exponent p (resp. p^2) for $p > 2$ (resp. $p = 2$) and order p^{3+2s_1} . More precisely, G is a central extension of its center $Z(G) = G'$ by the elementary abelian p -group $V = Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2}) = Z(\text{Ad}_{f_3})$:*

$$0 \longrightarrow Z(G) = G' \simeq (\mathbb{Z}/p\mathbb{Z})^3 \longrightarrow G \xrightarrow{\pi} Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2}) = Z(\text{Ad}_{f_3}) \simeq (\mathbb{Z}/p\mathbb{Z})^{2s_1} \longrightarrow 0$$

Furthermore, G is a p -Sylow subgroup of A , which is normal except when C is birational to the Hermitian curve: $W^q - W = X^{1+q}$, with $q = p^3$.

2. *There exists a coordinate X for the projective line C/G_2 , $s \geq 2$, $d \geq 2$ dividing s , and γ_2, γ_3 in $\mathbb{F}_{p^d} - \mathbb{F}_p$ linearly independent over \mathbb{F}_p , $b_1 \in k$, $c_1 \in k$ such that:*

f_1	$f_1(X) = X S_1(X)$	with	$S_1(F) = \sum_{j=0}^{s/d} a_{jd} F^{jd} \in k\{F\}$	$a_s = 1$
f_2	$f_2(X) = X S_2(X) + b_1 X$	with	$S_2 = \gamma_2 S_1$	
f_3	$f_3(X) = X S_3(X) + c_1 X$	with	$S_3 = \gamma_3 S_1$	
V	$V = Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2}) = Z(\text{Ad}_{f_3})$			

Therefore, the solutions can be parametrized by $s+4$ algebraically independent variables over \mathbb{F}_p , namely the s coefficients of S , $\gamma_2 \in \mathbb{F}_{p^d} - \mathbb{F}_p$, $\gamma_3 \in \mathbb{F}_{p^d} - \mathbb{F}_p$, $b_1 \in k$ and $c_1 \in k$.

Moreover,

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^s}{1+p+p^2} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+p+p^2)^2}$$

Proof: As $\ell_{1,2} = \ell_{2,3} = \ell_{1,3} = 0$, the second point of Lemma 5.11 first implies $v = 2s_3$. Applying [Ro08a] (Prop. 4.2), we gather that $s_1 = s_2 = s_3$, that $V = Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2}) = Z(\text{Ad}_{f_3})$ and we get the expected formulas for the functions f_i 's. Moreover, it follows from [Ro08a] (Prop. 4.3 and Rem. 4.5) that $G = A_{\infty,1}$ is a special group. The unicity of the p -Sylow subgroup is discussed in Remark 3.1. \square

5.4.3 Case: $[G', G] \supsetneq \text{Fratt}(G') = \{e\}$.

Lemma 5.13. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$ such that $[G', G] \neq \{e\}$. We keep the notations introduced in section 5.1. Then, one cannot have $\ell_{1,2} = \ell_{2,3} = 0$.*

Proof: Assume that $\ell_{1,2} = 0$ and $\ell_{2,3} = 0$. Since the representation ρ is non trivial, $\ell_{1,3} \neq 0$. The second point of Lemma 5.9 shows that $m_1 = m_2 = 1 + p^s$ with $s \geq 2$. The third point of Lemma 5.11 implies that $m_3 = 1 + 2p^s$ with $p \geq 3$ and $s \in \{1, 2\}$. Moreover, $v = s + 1$. As $s \geq 2$, we obtain: $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{(p+1)^2 p^2}{(1+p+2p^2)^2} < \frac{4}{(p^2-1)^2}$, hence a contradiction. As a conclusion, either $\ell_{1,2} \neq 0$ or $\ell_{2,3} \neq 0$. \square

As a consequence, there are 3 cases to study:

$\ell_{1,2} \neq 0$ and $\ell_{2,3} = 0$ (cf. Proposition 5.14).

$\ell_{1,2} = 0$ or $\ell_{2,3} \neq 0$ (cf. Proposition 5.15).

$\ell_{1,2} \neq 0$ or $\ell_{2,3} \neq 0$ (cf. Proposition 5.16).

Proposition 5.14. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$ such that $[G', G] \neq \{e\}$. We keep the notations introduced in section 5.1. Assume that $\ell_{1,2} \neq 0$ and $\ell_{2,3} = 0$.*

1. *Then, $p \geq 5$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i 's can be parametrized as follows:*

f_1	$f_1(X) = X^{1+p} + a_2 X^2$
V	$V = Z(\text{Ad}_{f_1}) = Z(X^{p^2} + 2a_2^p X^p + X)$
f_2	$f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_3 X^3 + b_1 X$
b_{1+2p}	$b_{1+2p} \in k^\times$
a_2	$2a_2^p = -b_{1+2p}^{-p} (b_{1+2p} + b_{1+2p}^{p^2}) \Leftrightarrow b_{1+2p} \in V$
V	$V = Z(X^{p^2} - b_{1+2p}^{-p} (b_{1+2p} + b_{1+2p}^{p^2}) X^p + X)$
b_{2+p}	$b_{2+p} = -b_{1+2p}^p$
b_3	$3b_3^p = b_{1+2p}^{-p} (b_{1+2p}^{2p^2} - b_{1+2p}^2)$
b_1	$b_1 \in k$
$\ell_{1,2}$	$\ell_{1,2}(y) = 2(b_{1+2p} y^p - b_{1+2p}^p y)$
f_3	$f_3(X) = c_{1+2p} X^{1+2p} + c_{2+p} X^{2+p} + c_3 X^3 + c_1 X$
c_{1+2p}	$c_{1+2p} \in k^\times$
c_{1+2p}	$c_{1+2p} \in V$, c_{1+2p} and b_{1+2p} \mathbb{F}_p -independent
c_{2+p}	$c_{2+p} = -c_{1+2p}^p$
c_3	$3c_3^p = -c_{1+2p}^{-p} (c_{1+2p}^{2p^2} + c_{1+2p}^2)$
c_1	$c_1 \in k$
$\ell_{1,3}$	$\ell_{1,3}(y) = 2(c_{1+2p} y^p - c_{1+2p}^p y)$
$\ell_{2,3}$	$\ell_{2,3}(y) = 0$

Therefore, the solutions are parametrized by 4 algebraically independent variables over \mathbb{F}_p , namely $b_{1+2p} \in k^\times$, $c_{1+2p} \in k^\times$, $b_1 \in k$ and $c_1 \in k$.

Moreover,

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^3}{1+2p+2p^2} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+2p+2p^2)^2}$$

2. In this case, $G = A_{\infty,1}$ is the unique p -Sylow subgroup of A .

Proof:

- Lemma 5.9 first shows that $m_1 = 1 + p^s$, $m_2 = 1 + 2p^s$, with $p \geq 3$ and $s \in \{1, 2\}$. Moreover, $v = s + 1$. As $\ell_{1,2} \neq 0$ and $\ell_{2,3} = 0$, the second point of Lemma 5.11 imposes $\ell_{1,3} \neq 0$. Then, Lemma 5.11 shows that $m_3 = 1 + 2p^s$. If $s = 2$, $m_1 = 1 + p^2$, $m_2 = m_3 = 1 + 2p^3$ and $v = 3$. So $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2(p+1)^2}{(1+2p+2p^2)^2} < \frac{4}{(p^2-1)^2}$, which contradicts condition (*). It follows that $s = 1$. In this case, $m_1 = 1 + p$, $m_2 = m_3 = 1 + 2p$, $v = 2$ and $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+2p+2p^2)^2}$. Therefore, condition (*) is satisfied as soon as $p \geq 5$. The parametrization of the functions f_i 's then derives from Proposition 5.6. Furthermore, the third condition (cf. Recall 4.2.1-c) imposed on the degree of the functions f_i 's requires that the parameters b_{1+2p} and c_{1+2p} are linearly independent over \mathbb{F}_p .
- The equality $G = A_{\infty,1}$ derives from the maximality of $V = Z(\text{Ad}_{f_1})$ (see Proposition 3.2). The unicity of the p -Sylow subgroup is due to Remark 3.1. \square

Proposition 5.15. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$ such that $[G', G] \neq \{e\}$. We keep the notations introduced in section 5.1. Assume that $\ell_{1,2} = 0$ and $\ell_{2,3} \neq 0$.*

- Then, $p \geq 5$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i 's can be parametrized as follows:

f_1	$f_1(X) = X^{1+p^2} + a_2 X^2$
f_2	$f_2(X) = \gamma_2 (X^{1+p^2} + a_2 X^2) + b_1 X$
b_1	$b_1 \in k$
γ_2	$\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$
V	$V = Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2}) = Z(X^{p^4} + 2a_2^{p^2} X^{p^2} + X)$

First case: $b_1 \neq 0$

f_3	$f_3(X) = c_{1+2p^2} X^{1+2p^2} + c_{2+p^2} X^{2+p^2} + c_{1+p^2} X^{1+p^2} + c_{1+p} X^{1+p} + c_3 X^3 + c_2 X^2 + c_1 X$
c_{1+2p^2}	$c_{1+2p^2} \in k^\times$
a_2	$2a_2^{p^2} = -c_{1+2p^2}^{-p^2} (c_{1+2p^2}^{p^4} + c_{1+2p^2}) \Leftrightarrow c_{1+2p^2} \in V$
V	$V = Z(X^{p^2} - c_{1+2p^2}^{-p} (c_{1+2p^2} + c_{1+2p^2}^{p^2}) X^p + X)$
c_{2+p^2}	$c_{2+p^2} = -c_{1+2p^2}^{p^2}$
c_3	$3c_3^{p^2} = -c_{1+2p^2}^{p^2} (3c_{1+2p^2}^{2p^4} + 4c_{1+2p^2}^{1+p^4} + c_{1+2p^2}^2)$
$e := c_{1+p^2} - c_{1+p^2}^{p^2}$	$e \in Z((c_{1+2p^2}^{p^7-p^3} + 1 + c_{1+2p^2}^{p-p^5} + c_{1+2p^2}^{p^7+p-p^5-p^3}) X^{1+p^4} - X^{1+p^2} - X^{p^2} - X - 1)$
b_1	$b_1^{p^5-p^4+p^3-p^2} = -e^{p^3-1}$
c_{1+p}	$c_{1+p}^{p+p^3} = -e^{1+p}$
c_2	$4c_2^{p^3(p-1)^2(p^2+1)} = c_{1+p^2}^{p^3(p^2+1)} + (c_{1+2p^2}^{p^7-p^3} + 1 + c_{1+2p^2}^{p-p^5} + c_{1+2p^2}^{p^7+p-p^5-p^3}) (c_{1+p^2} - c_{1+p^2}^{p^2})^{p^3+p^2+p-1}$
c_1	$c_1 \in k$

Therefore, the solutions can be parametrized by 3 algebraically independent variables over \mathbb{F}_p , namely $c_{1+2p^2} \in k^\times$, $c_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$. One also finds a fourth parameter $e := c_{1+p^2} - c_{1+p^2}^{p^2}$ which runs over the set of zeroes of a polynomial whose coefficients are rational functions in c_{1+2p^2} . So, for a given c_{1+2p^2} , the parameter e takes a finite number of values.

Second case: $b_1 = 0$

f_3	$f_3(X) = c_{1+2p^2} X^{1+2p^2} + c_{2+p^2} X^{2+p^2} + c_3 X^3$
c_{1+2p^2}	$c_{1+2p^2} \in k^\times$
a_2	$2a_2^{p^2} = -c_{1+2p^2}^{-p^2} (c_{1+2p^2}^{p^4} + c_{1+2p^2}) \Leftrightarrow c_{1+2p^2} \in V$
V	$V = Z(X^{p^2} - c_{1+2p^2}^{-p} (c_{1+2p^2} + c_{1+2p^2}^{p^2}) X^p + X)$
c_{2+p^2}	$c_{2+p^2} = -c_{1+2p^2}^{p^2}$
c_3	$3c_3^{p^2} = -c_{1+2p^2}^{p^2} (3c_{1+2p^2}^{2p^4} + 4c_{1+2p^2}^{1+p^4} + c_{1+2p^2}^2)$
c_1	$c_1 \in k$

In this case, the solutions can be parametrized by 3 algebraically independent variables over \mathbb{F}_p , namely $c_{1+2p^2} \in k^\times$, $c_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$.

In both cases,

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^4}{1+p+2p^2} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+p+2p^2)^2}$$

2. Moreover, $G = A_{\infty,1}$ is the unique p -Sylow subgroup of A .

Proof:

1. (a) We describe f_1 , f_2 and V .

Lemma 5.9 first implies that $m_1 = m_2 = 1 + p^s$, with $s \geq 2$. More precisely, we deduce from Proposition 5.5 that $f_1(X) = X S_1(X)$ and $f_2(X) = \gamma_2 X S_1(X) + b_1 X$, where S_1 is a monic additive polynomial with degree s in F , $b_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^d} - \mathbb{F}_p$ with d an integer dividing s . Moreover, $v = 2s$ and $V = Z(\text{Ad}_{f_1}) = Z(\text{Ad}_{f_2})$.

(b) We show that $\ell_{1,3} \neq 0$.

Indeed, assume that $\ell_{1,3} = 0$. Then, we deduce from Remark 5.10 that $m_3 = 1 + 2p^s$, with $s \in \{2, 3\}$ and $p \geq 3$. Moreover, $v = s + 1$. As $s \neq 1$, it follows that $s = 2$ and $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2(p+1)^2}{(1+p+2p^2)^2} < \frac{4}{(p^2-1)^2}$, which contradicts condition (*).

- (c) *We show that $f_3 \notin \Sigma_2$.*
 If $f_3 \in \Sigma_2$, the representation ρ is trivial, hence a contradiction. Therefore, $f_3 \notin \Sigma_2$ and one can define an integer $a \leq m_3$ such that X^a is the monomial of f_3 with highest degree among those that do not belong to Σ_2 . Since f_3 is assumed to be reduced mod $\wp(k[X])$, then $a \not\equiv 0 \pmod p$.
- (d) *We show that p divides $a - 1$.*
 Consider the equation:
- $$\forall y \in V, \quad \Delta_y(f_3) = \ell_{1,3}(y) f_1(X) + \ell_{2,3}(y) f_2(X) \quad \text{mod } \wp(k[X]) \quad (10)$$
- where $\ell_{1,3}$ and $\ell_{2,3}$ are non zero linear forms from V to \mathbb{F}_p . The monomials of f_3 with degree strictly lower than a belong to Σ_2 . So they give linear contributions in $\Delta_y(f_3) \text{ mod } \wp(k[X])$ (cf. [Ro08a] Lemma 3.9). Assume that p does not divide $a - 1$. Then, for all y in V , equation (10) gives the following equality mod $\wp(k[X])$:
- $$c_a(f_3) a X^{a-1} + \text{lower degree terms} = (\ell_{1,3}(y) + \gamma_2 \ell_{2,3}(y)) X^{1+p^s} + \text{lower degree terms}$$
- where $c_a(f_3) \neq 0$ denotes the coefficient of X^a in f_3 . If $a - 1 > 1 + p^s$, then $y = 0$ for all y in V and $V = \{0\}$ which is excluded for a big action (cf. [MR08] Prop. 2.2). If $a - 1 < 1 + p^s$, then, $\ell_{1,3}(y) + \gamma_2 \ell_{2,3}(y) = 0$, for all y in V . It follows that $\gamma_2 \in \mathbb{F}_p$, which is another contradiction. So, $a - 1 = 1 + p^s$ and by equating the corresponding coefficients in (10), one gets: $ay = \ell_{1,3}(y) + \gamma_2 \ell_{2,3}(y)$, for all y in V . So, $V \subset \mathbb{F}_p + \gamma_2 \mathbb{F}_p$ and $v \leq 2$. As $v = 2s$, we deduce that $s = 1$, which is a contradiction. Thus, p divides $a - 1$ and one can write $a = 1 + \lambda p^t$ with $t \geq 1$ and $\lambda \geq 2$, because of the definition of a . We also define $j_0 := a - p^t$.
- (e) *We show that $v \geq t + 1$.*
 By [Ro08a] (Lemma 3.11), $p^v \geq m_3 + 1 > m_3 - 1 \geq a - 1 = \lambda p^t \geq 2p^t$. This implies $v \geq t + 1$.
- (f) *We show that $j_0 = 1 + p^s$.*
 If $j_0 < 1 + p^s$, we gather the same contradiction as the one found in [Ro08a] [proof of Theorem 5.6, point 4, with $i = 2$]. Now, assume that $j_0 > 1 + p^s$. As in [MR08] [proof of Theorem 5.1, point 6], we prove that the coefficient of X^{j_0} in the left-hand side of (10) is $T(y)$, where T is a polynomial of $k[X]$ with degree p^t . If $j_0 > 1 + p^s$, then $T(y) = 0$, for all y in V . This implies $V \subset Z(T)$ and $v \leq t$, which contradicts the previous point.
- (g) *We show that either $v = t + 1$ or $v = t + 2$.*
 We have already seen that $v \geq t$. As $j_0 = 1 + p^s$, we equate the corresponding coefficients in (10) and obtain $T(y) = \ell_{1,3}(y) + \gamma_2 \ell_{2,3}(y)$, for all y in V . As $\ell_{1,3}(y) \in \mathbb{F}_p$ and $\ell_{2,3}(y) \in \mathbb{F}_p$, we get $T(y)^p - T(y) = \ell_{2,3}(y) (\gamma_2^p - \gamma_2)$, with $\gamma_2 \notin \mathbb{F}_p$. Then, for all y in V , $R(y) := \frac{T(y)^p - T(y)}{\gamma_2^p - \gamma_2} = \ell_{2,3}(y) \in \mathbb{F}_p$ and $V \subset Z(R^p - R)$. In particular, $v \leq t + 2$.
- (h) *We show that $m_3 = a = 1 + p^s + p^t$.*
 Assume that $m_3 > a$. Then, by definition of a , $m_3 = 1 + p^{s_3}$ with $s_3 \geq s$. Note that $s_3 \geq s + 1$. Otherwise, $m_3 = 1 + p^s = j_0 < a$. On the one hand, $|G| = p^{3+v} = p^{3+2s}$. On the other hand,
- $$g = \frac{p-1}{2} (p^s + p^{s+1} + p^2(m_3 - 1)) = \frac{p-1}{2} p^s (1 + p + p^{2+s_3-s}) \geq \frac{p-1}{2} p^s (1 + p + p^3)$$
- Thus, $\frac{|G|}{g^2} \leq \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+p+p^3)^2} < \frac{4}{(p^2-1)^2}$. This contradicts condition (*), so $m_3 = a$.
- (i) *We show that $s = 2$ and $v = 4$. In particular, $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$.*
 We already know that $s \geq 2$ and $v = 2s \geq 4$. So, $|G| = p^{3+v} \leq p^7$. Assume that $s \geq 3$. Then, as $t \geq 1$, we get: $g = \frac{p-1}{2} (p^s + p^{s+1} + p^2(m_3 - 1)) = \frac{p-1}{2} (p^s + p^{s+1} + p^{s+2} + p^{t+2}) \geq \frac{p-1}{2} (2p^3 + p^4 + p^5)$. It follows that $\frac{|G|}{g^2} \leq \frac{4}{(p^2-1)^2} \frac{p(p+1)^2}{(2+p+p^2)^2} < \frac{4}{(p^2-1)^2}$, which is a contradiction. So $s = 2$ and $v = 4$. We have previously mentioned that $\gamma_2 \in \mathbb{F}_{p^d} - \mathbb{F}_p$, where d is an integer dividing s . As $s = 2$, the only possibility is $d = 2$.
- (j) *We deduce that $t = s = 2$, so $m_3 = 1 + 2p^2$ and $p \geq 5$.*
 We have seen $v = t + 1$ or $v = t + 2$, with $t \geq 1$. As $v = 4$, there are two possibilities either $t = 2$ or $t = 3$. If $t = 3$, $|G| = p^7$ and $g = \frac{p-1}{2} p^2 (1 + p + 2p^3)$. So, $\frac{|G|}{g^2} \leq \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+p+2p^3)^2} < \frac{4}{(p^2-1)^2}$. Therefore, $t = 2 = s$. In this case, $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+p+2p^2)^2}$ and condition (*) requires $p \geq 5$.

(k) We gather the parametrization of f_1, f_2 and V .

As $s = d = 2$, f_1 reads $f_1(X) = X S_1(X)$ with $S_1(F) = \sum_j^{s/d} a_{jd} F^{jd} = a_0 I + F^2$, since S_1 is assumed to be monic. Then,

$$f_1(X) = X(X^{p^2} + a_2 X^2) \quad \text{and} \quad f_2(X) = \gamma_2 X(X^{p^2} + a_2 X^2) + b_1 X$$

with $a_2 \in k$, $b_1 \in k$ and $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$. In this case,

$$V = Z(Ad_{f_1}) = Z(X^{p^4} + 2a_2^{p^2} X^{p^2} + X)$$

(l) We show that $f_3 \in \Sigma_4$ but $f_3 \notin \Sigma_4 - \Sigma_3$.

By [Ro08a] (Thm. 3.13), $f_3 \in \Sigma_4$. We now show that f_3 does not have any monomial in $\Sigma_4 - \Sigma_3$. Indeed, as $m_3 = 1 + 2p^2$, the possible monomials of f_3 in $\Sigma_4 - \Sigma_3$ are X^{1+2p+p^2} , X^{2+p+p^2} , X^{3+p^2} , X^{1+3p} , X^{2+2p} , X^{3+p} and X^4 . Now, equate the coefficients of the monomial $X^{1+p+p^2} \in \Sigma_3$ in each side of (10). In the left-hand side, i.e. in $\Delta_y(f_3) \bmod \wp(k[X])$, X^{1+p+p^2} is produced by monomials X^b of f_3 that belong to $\Sigma_4 - \Sigma_3$ and satisfy $b > 1 + p + p^2$. This leaves only two possibilities: X^{1+2p+p^2} and X^{2+p+p^2} . In the right-hand side of (10), $X^{1+p+p^2} \in \Sigma_3 - \Sigma_2$ does not occur since $\ell_{1,3}(y) f_1(X) + \ell_{2,3}(y) f_2(X)$ lies in Σ_2 . It follows that, for all y in V , $2c_{2+p+p^2} y^p + 2c_{1+2p+p^2} y = 0$, where c_t denotes the coefficient of the monomial X^t in f_3 . As $v = \dim_{\mathbb{F}_p} V = 4$, we deduce that $c_{2+p+p^2} = c_{1+2p+p^2} = 0$. We go on this way and equate successively the coefficients of X^{2+p^2} , X^{1+2p} , X^{2+p} and X^3 to prove that f_3 does not contain any monomial in $\Sigma_4 - \Sigma_3$. Therefore, f_3 reads as follows:

$$\begin{aligned} f_3(X) = & c_{1+2p^2} X^{1+2p^2} + c_{1+p+p^2} X^{1+p+p^2} + c_{2+p^2} X^{2+p^2} + c_{1+p^2} X^{1+p^2} + \\ & c_{1+2p^2} X^{1+2p} + c_{2+p} X^{2+p} + c_{1+p} X^{1+p} + c_3 X^3 + c_2 X^2 + c_1 X \end{aligned}$$

(m) We determine f_3 .

We finally have to solve (10) with f_1, f_2 and f_3 as described above. Calculation show that $c_{1+p+p^2} = c_{1+2p^2} = c_{2+p} = 0$ and that the coefficients a_2, c_{2+p^2} and c_3 can be expressed as rational functions in c_{1+2p^2} (see formulas in the table given in the proposition). To conclude, one has to distinguish the cases $b_1 \neq 0$ and $b_1 = 0$. In the first case, b_1, c_{1+p} and c_2 can be expressed as rational functions in c_{1+p^2} whereas $e := c_{1+p^2} - c_{1+p^2}^{p^2}$ belongs to the set of zeroes of a polynomial whose coefficients are rational functions in c_{1+2p^2} (see table). When $b_1 = 0$, then $c_{1+p} = 0$, $c_1 = c_{1+p^2} a_2$ and $c_{1+p^2} \in \mathbb{F}_{p^2}$. It follows that $f_3(X) = c_{1+2p^2} X^{1+2p^2} + c_{2+p^2} X^{2+p^2} + c_3 X^3 + c_1 X + c_{1+p^2} f_1(X)$. As $\gamma_2 \in \mathbb{F}_{p^2} - \mathbb{F}_p$, $\{1, \gamma_2\}$ is a basis of \mathbb{F}_{p^2} over \mathbb{F}_p . Write $\epsilon = \epsilon_1 + \epsilon_2 \gamma_2$, with ϵ_1 and ϵ_2 in \mathbb{F}_p . By replacing f_3 with $f_3 - (\epsilon_1 f_1 + \epsilon_2 f_2)$, one obtains the expected formula.

2. The equality $G = A_{\infty,1}$ derives from the maximality of $V = Z(Ad_{f_1})$ (see Proposition 3.2). The unicity of the p -Sylow subgroup is due to Remark 3.1. \square

The last case: $\ell_{1,2} \neq 0$ and $\ell_{2,3} \neq 0$, generalizes the results obtained in [Ro08a] (section 6.2).

Proposition 5.16. *Let (C, G) be a big action satisfying $\mathcal{G}_*^{p^3}$ such that $[G', G] \neq \{e\}$. We keep the notations introduced in section 5.1. Assume that $\ell_{1,2} \neq 0$ and $\ell_{2,3} \neq 0$.*

1. *Then, $p \geq 11$ and there exists a coordinate X for the projective line C/G_2 such that the functions f_i 's can be parametrized as follows:*

f_1	$f_1(X) = X^{1+p} + a_2 X^2$
V	$V = Z(\text{Ad}_{f_1}) = Z(X^{p^2} + 2a_2^p X^p + X)$
f_2	$f_2(X) = b_{1+2p} X^{1+2p} + b_{2+p} X^{2+p} + b_3 X^3 + b_1 X$
b_{1+2p}	$b_{1+2p} \in k^\times$
a_2	$2a_2^p = -b_{1+2p}^{-p} (b_{1+2p} + b_{1+2p}^{p^2}) \Leftrightarrow b_{1+2p} \in V$
V	$V = Z(X^{p^2} - b_{1+2p}^{-p} (b_{1+2p} + b_{1+2p}^{p^2}) X^p + X)$
b_{2+p}	$b_{2+p} = -b_{1+2p}^p$
b_3	$3b_3^p = b_{1+2p}^{-p} (b_{1+2p}^{2p^2} - b_{1+2p}^2)$
$\ell_{1,2}$	$\ell_{1,2}(y) = 2(b_{1+2p} y^p - b_{1+2p}^p y)$
f_3	$f_3(X) = c_{1+3p} X^{1+3p} + c_{2+2p} X^{2+2p} + c_{1+2p} X^{1+2p} + c_{3+p} X^{3+p}$ $+ c_{2+p} X^{2+p} + c_{1+p} X^{1+p} + c_4 X^4 + c_3 X^3 + c_2 X^2 + c_1 X$
c_{1+3p}	$3c_{1+3p} = 2b_{1+2p}^2$
c_{2+2p}	$c_{2+2p} = -b_{1+2p}^{1+p}$
c_{3+p}	$3c_{3+p} = 2b_{1+2p}^{2p}$
c_4	$6c_4^p = -b_{1+2p}^{-p} (b_{1+2p}^3 + b_{1+2p}^{3p^2})$
c_{1+2p}	$c_{1+2p} \in V$
c_{2+p}	$c_{2+p} = -c_{1+2p}^p$
c_3	$3c_3^p = b_{1+2p}^{-p} (b_{1+2p} + b_{1+2p}^{p^2}) (c_{1+2p}^{p^2} - c_{1+2p})$
c_{1+p}	$c_{1+p} \in k$
b_1	$2b_1^p = b_{1+2p}^{-p} (c_{1+p}^p - c_{1+p})$
c_2	$2c_2^p = -b_{1+2p}^{-p} (c_{1+p}^p b_{1+2p}^{p^2} + c_{1+p} b_{1+2p})$
c_1	$c_1 \in k$
$\ell_{1,3}$	$\ell_{1,3}(y) = 2(c_{1+2p} y^p c_{1+p}^p - y) + 2b_{1+2p}^2 y^{2p} - 4b_{1+2p}^{1+p} y^{1+p} + 2b_{1+2p}^{2p} y^2$ $= 2(c_{1+2p} y^p - c_{1+p}^p y) + \ell_{1,2}(y)/2$
$\ell_{2,3}$	$\ell_{2,3}(y) = 2(b_{1+2p} y^p - b_{1+2p}^p y)$

Therefore, the solutions can be parametrized by 3 algebraically independent variables over \mathbb{F}_p , namely $b_{1+2p} \in k^\times$, $c_{1+p} \in k$ and $c_1 \in k$. One also finds a fourth parameter c_{1+2p} which runs over V . So, for a given b_{1+2p} , the parameter c_{1+2p} takes a finite number of values. Moreover,

$$\frac{|G|}{g} = \frac{2p}{p-1} \frac{p^3}{1+2p+3p^2} \quad \text{and} \quad \frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+2p+3p^2)^2}$$

2. $G = A_{\infty,1}$ is the unique p -Sylow subgroup of A . Furthermore, $Z(G)$ is cyclic of order p .

Proof:

1. In this case, the group G satisfies the third condition of [Ro08a] (Prop. 5.2). So, we deduce from [Ro08a] (Thm. 5.6) that $m_1 = 1 + p^s$, $m_2 = 1 + 2p^s$, $m_3 = 1 + 3p^s$ with $p \geq 5$ and $v = s + 1$. Furthermore, it follows from Lemma 5.9 that $s \in \{1, 2\}$. Assume that $s = 2$. Then, $|G| = p^6$, $g = \frac{p-1}{2} p^2 (1 + 2p + 3p^2)$, so $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^2(p+1)^2}{(1+2p+3p^2)^2} < \frac{4}{(p^2-1)^2}$. This is a contradiction, hence $s = 1$. In this case, $\frac{|G|}{g^2} = \frac{4}{(p^2-1)^2} \frac{p^3(p+1)^2}{(1+2p+3p^2)^2}$ and condition (*) is satisfied as soon as $p \geq 11$. Then, we deduce from Proposition 5.6 the parametrization of f_1 , V and f_2 mentioned in the table. Besides, we deduce from [Ro08a] (Thm. 5.6) that f_3 is in $\Sigma_4 - \Sigma_3$ with $m_3 = 1 + 3p$. This means that f_3 reads as follows:

$$f_3(X) = c_{1+3p} X^{1+3p} + c_{2+2p} X^{2+2p} + c_{1+2p} X^{1+2p} + c_{3+p} X^{3+p}$$

$$+ c_{2+p} X^{2+p} + c_{1+p} X^{1+p} + c_4 X^4 + c_3 X^3 + c_2 X^2 + c_1 X$$

We determine the expressions of the coefficient by solving the equation:

$$\forall y \in V, \quad \Delta_y(f_3) = \ell_{1,3}(y) f_1(X) + \ell_{2,3} f_2(X) \quad \text{mod } \wp(k[X])$$

with $\ell_{1,2}(y) = \ell_{2,3}(y) = 2(b_{1+2p} y^p - b_{1+2p}^p y)$ (cf. [Ro08a], Prop. 5.4.1). The results are gathered in the table above.

2. The equality $G = A_{\infty,1}$ derives from [MR08] (Cor. 5.7). The unicity of the p -Sylow subgroup comes from Remark 3.1. The description of the center is due to [MR08] (Prop. 6.15). \square

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